ON THE GLOBAL WELL-POSEDNESS OF 2-D DENSITY-DEPENDENT NAVIER-STOKES SYSTEM WITH VARIABLE VISCOSITY

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ABSTRACT. Given solenoidal vector $u_0 \in H^{-2\delta} \cap H^1(\mathbb{R}^2)$, $\rho_0 - 1 \in L^2(\mathbb{R}^2)$, and $\rho_0 \in L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2)$ with a positive lower bound for $\delta \in (0, \frac{1}{2})$ and $2 < r < \frac{2}{1-2\delta}$, we prove that 2-D incompressible inhomogeneous Navier-Stokes system (1.1) has a unique global solution provided that the viscous coefficient $\mu(\rho_0)$ is close enough to 1 in the L^∞ norm compared to the size of δ and the norms of the initial data. With smoother initial data, we can prove the propagation of regularities for such solutions. Furthermore, for $1 < p < 4$, if $(\rho_0 - 1, u_0)$ belongs to the critical Besov spaces $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \times \big(\dot{B}^{-1+\frac{2}{p}}_{p,1} \cap L^2(\mathbb{R}^2)\big)$ and the $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ norm of $\rho_0 - 1$ is sufficiently small compared to the exponential of $\|u_0\|_{L^2}^2 + \|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}$, we prove the global well-posedness of (1.1) in the scaling invariant spaces. Finally for initial data in the almost critical Besov spaces, we prove the global well-posedness of (1.1) under the assumption that the L^∞ norm of $\rho_0 - 1$ is sufficiently small.

Keywords: Inhomogeneous Navier-Stokes systems, Littlewood-Paley Theory, critical regularity

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1. Introduction

The purpose of this paper is to investigate the global well-posedness of the following two-dimensional incompressible inhomogeneous Navier-Stokes equations with variable viscous coefficient

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) d) + \nabla \Pi = 0, \\
\operatorname{div} u = 0, \\
\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0,
\end{cases} \tag{1.1}
$$

where $\rho, u = (u_1, u_2)$ stand for the density and velocity of the fluid respectively, and $d = \big(\frac{1}{2}(\partial_i u_j + \partial_j u_i)\big)_{2\times 2}$ denotes the deformation tensor, Π is a scalar pressure function, and the viscous coefficient $\mu(\rho)$ is a smooth, positive and non-decreasing function on $[0, \infty)$. Such a system describes for instance a fluid that is incompressible but has nonconstant density owing to the complex structure of the flow due to a mixture (e.g. blood flow) or pollution (e.g. model of rivers). It may also describe a fluid containing a melted substance.

When $\mu(\rho)$ is a positive constant, and the initial density has a positive lower bound, Ladyženskaja and Solonnikov [19] first addressed the question of unique solvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u. Under the assumptions that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ $(p > d)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [19] proved

- Global well-posedness in dimension $d = 2$;
- Local well-posedness in dimension $d = 3$. If in addition u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds true.

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Danchin [11] proved similar well-posedness result of (1.1) in the whole space and with the initial data in the almost critical Sobolev spaces. In particular, in two space dimensions, he proved the the global well-poseness of (1.1) with $\mu(\rho)=\mu>0$ provided that the initial data (ρ_0,u_0) satisfies

$$\begin{aligned}&\rho_0-1\in H^{1+\alpha}(\mathbb{R}^2)\ (\nabla\rho_0\in L^\infty(\mathbb{R}^2)\text{ if }\alpha=1),\quad \rho_0\geq \underline{b}>0,\quad \text{and}\\ &u_0\in H^\beta(\mathbb{R}^2)\quad\text{for any}\quad \alpha>0,\quad \beta\in(0,\alpha)\cap(\alpha-1,\alpha+1).\end{aligned}$$

Very recently, Paicu, Zhang and Zhang [22] proved the global well-posedness of (1.1) with $\mu(\rho)=\mu>0$ for initial data: $\rho_0\in L^\infty(\mathbb{R}^2)$ with a positive lower bound and $u_0\in H^s(\mathbb{R}^2)$ for any $s>0$. This result improves the former interesting well-posedness theorem of Danchin and Mucha [14] by removing the smallness assumption on the fluctuation to the initial density and also with much less regularity for the initial velocity.

In general, Lions [20] (see also the references therein, and the monograph [5]) proved the global existence of finite energy weak solutions to (1.1). Yet the uniqueness and regularities of such weak solutions are big open questions even in two space dimensions, as was mentioned by Lions in [20] (see page 31-32 of [20]). Except under the additional assumptions that

$$\|\mu(\rho_0)-1\|_{L^\infty(\mathbb{T}^2)}\leq\varepsilon_0\quad\text{and}\quad u_0\in H^1(\mathbb{T}^2),\tag{1.2}$$

Desjardins [15] proved that the global weak solution, $(\rho,u,\nabla\Pi)$, constructed in [20] satisfies $u\in L^\infty((0,T);H^1(\mathbb{T}^2))$ and $\rho\in L^\infty((0,T)\times\mathbb{T}^2)$ for any $T<\infty$. Moreover, with additional regularity assumptions on the initial data, he could also prove that $u\in L^2((0,\tau);H^2(\mathbb{T}^2))$ for some short time τ (see Theorem 1.1 below).

To understand the system (1.1) further, the second author to this paper proved the global well-posedness to a modified 2-D model system, which coincides with the 2-D inhomogeneous Navier-Stokes equations with $\mu(\rho)=\mu>0$, with general initial data in [26]. Gui and Zhang [16] proved the global well-posedness of (1.1) with initial data satisfying $\|\rho_0-1\|_{H^{s+1}}$ being sufficiently small and $u_0\in H^s\cap\dot H^{-2\delta}(\mathbb{R}^2)$ for some $s>2$ and $0<\delta<\frac12$. Yet the exact size of $\|\rho_0-1\|_{H^{s+1}}$ was not presented in [16]. Huang, Paicu and Zhang [17] basically proved that as long as

$$\eta\overset{\text{def}}{=}\|\rho_0-1\|_{B^{\frac2p}_{p,1}}\exp\Bigl\{C_0\bigl(1+\mu^2(1)\bigr)\exp\bigl(\frac{C_0}{\mu^2(1)}\|u_0\|^2_{B^{-1+\frac2p}_{p,1}}\bigr)\Bigr\}\leq\frac{c_0\mu(1)}{1+\mu(1)},\tag{1.3}$$

for some sufficiently small c_0, (1.1) has a global solution so that $\rho-1\in\mathcal{C}_b([0,\infty);B^{\frac2p}_{p,1}(\mathbb{R}^2))$ and $u\in\mathcal{C}_b([0,\infty);B^{-1+\frac2p}_{p,1}(\mathbb{R}^2))\cap L^1(\mathbb{R}^+;B^{1+\frac2p}_{p,1}(\mathbb{R}^2))$ for $1<p<4$. In a recent preprint [18], Huang and Paicu can prove the global existence of solution of (1.1) with much weaker assumption than (1.3). Yet as there is no $L^1((0,T);Lip(\mathbb{R}^2))$ estimate for the velocity field, the uniqueness of such solutions is not clear in [18].

Let $\mathcal{R}$ be the usual Riesz transform, $\mathcal{Q}\overset{\text{def}}{=}\nabla(\Delta)^{-1}\mathrm{div}$, and $\mathbb{P}\overset{\text{def}}{=}I-\mathcal{Q}$ be the Leray projection operator on the space of divergence-free vector fields, we first recall the following result from Desjardins [15]:

Theorem 1.1. *Let $\rho\in L^\infty(\mathbb{T}^2)$, $u_0\in H^1(\mathbb{T}^2)$ with $\mathrm{div}\,u_0=0$. Then there exists a positive constant ε_0 such that under the assumption of* (1.2), *Lions weak solutions* ([20]) *to* (1.1) *satisfy the following regularity properties for all $T>0$:*

- *$u\in L^\infty((0,T);\,H^1(\mathbb{T}^2))$ and $\sqrt{\rho}\partial_t u\in L^2((0,T)\times\mathbb{T}^2)$,*
- *ρ and $\mu(\rho)\in L^\infty((0,T)\times\mathbb{T}^2)\cap\mathcal{C}([0,T];\,L^p(\mathbb{T}^2))$ for all $p\in[1,\infty)$,*
- *$\nabla(\Pi-\mathcal{R}_i\mathcal{R}_j(2\mu d_{ij}))$ and $\nabla(\mathbb{P}\bigotimes\mathcal{Q}(2\mu d))_{ij}\in L^2((0,T)\times\mathbb{T}^2)$,*
- *Π may be renormalized in such a way that for some universal constant $C_0>0$,*

$$\Pi\quad\text{and}\quad\nabla u\in L^2((0,T);\,L^p(\mathbb{T}^2))\quad\text{for all }p\in[4,p^*],\tag{1.4}$$

where

$$\frac{1}{p^*} = 2C_0\|\mu(\rho_0) - 1\|_{L^\infty}. \tag{1.5}$$

Moreover, if $\mu(\rho_0) \geq \underline{\mu}$ *and* $\log(\mu(\rho_0)) \in W^{1,r}(\mathbb{T}^2)$ *for some* $r > 2$, *there exists some positive time* τ *so that* $u \in L^2((0,\tau); H^2(\mathbb{T}^2))$ *and* $\mu(\rho) \in C([0,\tau]; W^{1,\bar{r}}(\mathbb{T}^2))$ *for any* $\bar{r} < r$.

In what follows, we shall always assume that

$$0 < \underline{\mu} \leq \mu(\rho_0), \quad \mu(\cdot) \in W^{2,\infty}(\mathbb{R}^+) \quad \text{and} \quad \mu(1) = 1.$$

Notations: In the rest of this paper, we always denote a_+ to be any number strictly bigger than a and a_- any number strictly less then a. We shall denote $[Y]$ the integer part of Y, and $\bar{C}$ to be a uniform constant depending only m, M in (1.6) below and $\|\mu'\|_{L^\infty}$, which may change from line to line.

Our first purpose in this paper is to prove the following global well-posedness result for (1.1).

Theorem 1.2. *Let* m, M *be two positive constants and* $\delta \in (0, \frac{1}{2})$, $2 < r < \frac{2}{1-2\delta}$. *Let* $u_0 \in H^{-2\delta} \cap H^1(\mathbb{R}^2)$ *be a solenoidal vector filed, and* $\rho_0 - 1 \in L^2 \cap L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2)$ *satisfy*

$$m \leq \rho_0 \leq M, \quad \|\mu(\rho_0) - 1\|_{L^\infty} \leq \varepsilon_0, \tag{1.6}$$

and for some $q \in (1/\delta, p^*]$,

$$C_0 \stackrel{\text{def}}{=} \|u_0\|_{H^{-2\delta}}^2 + \|\rho_0 - 1\|_{L^2}^4 + \|u_0\|_{L^2}^4 + \|\nabla u_0\|_{L^2}^2 \exp\bigl(C\|u_0\|_{L^2}^4\bigr). \tag{1.7}$$

there holds

$$\|\mu(\rho_0) - 1\|_{L^\infty}\Bigl(\frac{1}{\delta} + 4\bar{C}^2C_0\bigl(1 + \|\rho_0\|_{B^{(2/q)_+}_{\infty,\infty}}\bigr)\exp\bigl(\bar{C}C_0\bigr)\Bigr) \leq \varepsilon_0 \tag{1.8}$$

for some sufficiently small ε_0. *Then* (1.1) *has a unique global solution* $(\rho, u, \nabla\Pi)$ *with* $\rho - 1 \in C_b([0,\infty); L^2 \cap L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2))$, $u \in \mathcal{C}_b([0,\infty); H^1(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; H^2(\mathbb{R}^2))$, $\partial_t u, \nabla\Pi \in L^1(\mathbb{R}^+; L^2(\mathbb{R}^2))$, *and*

$$\|\nabla u\|_{L^1(\dot{B}^0_{\infty,1})} \leq 2\bar{C}C_0\bigl(1 + \|\rho_0\|_{B^{(2/q)_+}_{\infty,\infty}}\bigr)\exp\bigl(\bar{C}C_0\bigr). \tag{1.9}$$

If in addition, $\mu(\cdot) \in W^{2+[s],\infty}(\mathbb{R}^+)$, $\rho_0 - 1 \in H^{1+s}(\mathbb{R}^2)$ *and* $u_0 \in H^s(\mathbb{R}^2)$ *for some* $s > 1$, *then the global solution* $\rho - 1 \in \mathcal{C}([0,\infty); H^{1+s}(\mathbb{R}^2))$, $u \in \mathcal{C}([0,\infty); H^s(\mathbb{R}^2)) \cap \widetilde{L}^1_{loc}(\mathbb{R}^+; \dot{H}^{2+s}(\mathbb{R}^2))$.

Remark 1.1. *Without the assumptions that* $\rho_0 - 1 \in L^2(\mathbb{R}^2)$ *and* $u_0 \in H^{-2\delta}(\mathbb{R}^2)$ *in Theorem 1.2, our proof of Theorem 1.2 ensures that* (1.1) *has a unique solution* (ρ, u) *on a time interval* $[0, \mathfrak{T}]$ *with* $\mathfrak{T}$ *being determined by*

$$\mathfrak{T} \geq \bigl(C(m, M, \|\nabla\rho_0\|_{L^r}, \|u_0\|_{H^1})\|\mu(\rho_0) - 1\|_{L^\infty}\bigr)^{-1}$$

and $\rho \in L^\infty((0,\mathfrak{T}); L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2))$, $u \in L^\infty((0,\mathfrak{T}); H^1(\mathbb{R}^2)) \cap L^2((0,\mathfrak{T}); H^2(\mathbb{R}^2))$.

Remark 1.2. *We should point out that the reason why Desjardin* [15] *only proved* (1.4) *for* $p \in [2, p^*]$ *with* p^* *being determined by* (1.5) *is because of the fact that the Riesz transform* $\mathcal{R}$ *maps continuously from* $L^p(\mathbb{R}^d)$ *to* $L^p(\mathbb{R}^d)$ *with the operator norm (see Theorem 3.1.1. of* [8] *for instance):*

$$\|\mathcal{R}\|_{\mathcal{L}(L^p\to L^p)} \leq C_0 p$$

for some uniform constant C_0. *Our main observation used in the proof of Theorem 1.2 is that: Riesz transform* $\mathcal{R}$ *maps continuously from homogeneous Besov spaces* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *(see Definition A.1) to* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *with the operator norm*

$$\|\mathcal{R}\|_{\mathcal{L}(\dot{B}^s_{p,r}\to\dot{B}^s_{p,r})} \leq C_0,$$

which enables us to prove the a priori estimate for $\|\nabla u\|_{L^1_T(L^\infty)}$. *This is in fact the most important ingredient used in the proof of Theorem 1.2.*

The other important ingredient used in the proof of (1.9) *is the time decay estimates* (2.12) *and* (2.13)*, which is a slight generalization of the decay estimates obtained by Huang and Paicu in* [18]*. The proof of such decay estimates is a direct application of Schonbek's frequency splitting method as well as the strategy of Wiegner* [25] *to prove the time decay estimate for classical 2-D Navier-Stokes system.*

In the particular case when $\mu(\rho)$ is a positive constant, the proof of Theorem 1.2 yields the following corollary, which does not require any low frequency assumption on u_0.

Corollary 1.1. *Let $\alpha \in (0,1)$ and m, M be positive constants. Let $u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ be a solenoidal vector filed and $\rho_0 - 1 \in \dot{B}^1_{2,1} \cap \dot{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)$ with $m \leq \rho_0 \leq M$. Then* (1.1) *with $\mu(\rho) = 1$ has a unique global solution (ρ, u) so that $\rho - 1 \in \mathcal{C}([0,\infty); \dot{B}^1_{2,1} \cap \dot{B}^\alpha_{\infty,\infty}(\mathbb{R}^2))$, $u \in \mathcal{C}([0,\infty); \dot{B}^0_{2,1}(\mathbb{R}^2)) \cap L^1_{loc}(\mathbb{R}^+; \dot{B}^2_{2,1}(\mathbb{R}^2))$.*

Another important feature of (1.1) is the scaling invariant property, namely, if (ρ, u) is a solution of (1.1) associated to the initial data (ρ_0, u_0), then

$$(1.10)\quad \big(\rho_\lambda(t,x), u_\lambda(t,x)\big) \stackrel{\text{def}}{=} \big(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x)\big) \quad \big(\rho_{0,\lambda}(x), u_{0,\lambda}(x)\big) \stackrel{\text{def}}{=} \big(\rho_0(\lambda x), \lambda u_0(\lambda x)\big),$$

$(\rho_\lambda(t,x), u_\lambda(t,x))$ is a also a solution of (1.1) associated with the initial data $(\rho_{0,\lambda}(x), u_{0,\lambda}(x))$. A functional space for the data (ρ_0, u_0) or for the solution (ρ, u) is said to be at the scaling of the equation if its norm is invariant under the transformation (1.10). In the very interesting paper [13], Danchin and Mucha proved the global well-posedness of (1.1) with $\mu(\rho) = \mu > 0$ in d space dimensions and with small initial data in the critical spaces $\rho_0 - 1 \in \dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ for $p \in [1, 2d)$. In fact, they [13] only require $\rho_0 - 1$ to be small in the multiplier space of $\dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$. One may check [13] and the references therein for more details in this direction.

It is easy to check that $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \times \big(\dot{B}^{-1+\frac{2}{p}}_{p,1} \cap L^2(\mathbb{R}^2)\big)$ is at the scaling of (1.1). When $\rho_0 - 1$ is small enough in the critical space $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, we have the following global well-posedness result for (1.1), which in particular improves the smallness condition (1.3) in [17] to (1.11) below (with only one exponential), and completes the uniqueness gap for $p \in (2,4)$ in [17].

Theorem 1.3. *Let $1 < p < 4$, $\rho_0 - 1 \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ and $u_0 \in \dot{B}^{-1+\frac{2}{p}}_{p,1} \cap L^2(\mathbb{R}^2)$ which satisfy $\operatorname{div} u_0 = 0$ and*

$$(1.11)\qquad \|\rho_0 - 1\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \exp\big\{C_0(\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}} + \|u_0\|^2_{L^2})\big\} \leq \varepsilon_0$$

for some uniform constant C_0 and ε_0 being sufficiently small. Then (1.1) *has a unique global solution $(\rho, u, \nabla\Pi)$ so that $\rho \in \mathcal{C}_b([0,\infty); \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))$, $u \in \mathcal{C}_b([0,\infty); \dot{B}^{-1+\frac{2}{p}}_{p,1} \cap L^2(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \dot{B}^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$, and $\partial_t u, \nabla\Pi \in L^1(\mathbb{R}^+; \dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$.*

Finally, in the case when the initial data is in the almost scaling invariant spaces and $\|\rho_0 - 1\|_{L^\infty}$ is sufficiently small, we have the following global well-posedness result for (1.1):

Theorem 1.4. *Let $1 < p < 4$ and $0 < \varepsilon < \frac{4}{p} - 1$. Let $\rho_0 - 1 \in \dot{B}^{\frac{2}{p}}_{p,1} \cap \dot{B}^{\frac{2}{p}+\varepsilon}_{p,1}(\mathbb{R}^2)$ and $u_0 \in \dot{B}^{-1+\frac{2}{p}}_{p,1} \cap \dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1} \cap L^2(\mathbb{R}^2)$ be a solenoidal vector filed. Then* (1.1) *has a unique global solution $(\rho, u, \nabla\Pi)$ so that $\rho - 1 \in \mathcal{C}_b([0,\infty); \dot{B}^{\frac{2}{p}}_{p,1} \cap \dot{B}^{\frac{2}{p}+\varepsilon}_{p,1}(\mathbb{R}^2))$, $u \in \mathcal{C}_b([0,\infty); \dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1} \cap \dot{B}^{-1+\frac{2}{p}}_{p,1} \cap L^2(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \dot{B}^{1+\frac{2}{p}}_{p,1} \cap \dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1}(\mathbb{R}^2))$, and $\partial_t u, \nabla\Pi \in L^1(\mathbb{R}^+; \dot{B}^{-1+\frac{2}{p}}_{p,1} \cap \dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1}(\mathbb{R}^2))$ provided that*

$$(1.12)\qquad \|\rho_0 - 1\|_{L^\infty} \leq \varepsilon_0$$

for some small enough ε_0.

Remark 1.3. *One may check* (8.8) *for the exact size of* ε_0 *in* (1.12).

Scheme of the proof and organization of the paper. In the second section, we shall present the *a priori* time decay estimate for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$ which leads to the crucial estimate for $\|\nabla u\|_{L^1(\mathbb{R}^+;L^q)}$ for q satisfying $q\delta > 1$. Based on these estimates and the observation in Remark 1.2, in Section 3, we shall present the *a priori* $L^1(\mathbb{R}^+;\dot{B}^1_{\infty,1})$ estimate for velocity field. In Section 4, we present a blow-up criterion for smooth enough solutions of (1.1). We then present the proofs of Theorem 1.2 in Section 5 and Corollary 1.1 in Section 6. Finally we present the proofs of Theorem 1.3 in Section 7 and Theorem 1.4 in Section 8. For the convenience of the readers, we collect some basic facts on Littlewood-Paley analysis, which has been used throughout the paper, in the Appendix A.

Let us complete this section with the notations we are going to use in this context.

Notations: Let A, B be two operators, we denote $[A,B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ (or $(a|b)_{L^2}$) the $L^2(\mathbb{R}^2)$ inner product of a and b.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\,X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I;\,X)$ the subset of bounded functions of $\mathcal{C}(I;\,X)$. For $q \in [1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^q(I)$. For any vector field $v = (v_1, v_2)$, we denote $d(v) = \frac{1}{2}\big(\partial_i v_j + \partial_j v_i\big)_{i,j=1,2}$. Finally, $(d_j)_{j\in\mathbb{Z}}$ (resp. $(c_j)_{j\in\mathbb{Z}}$) will be a generic element of $\ell^1(\mathbb{Z})$ (resp. $\ell^2(\mathbb{Z})$) so that $\sum_{j\in\mathbb{Z}} d_j = 1$ (resp. $\sum_{j\in\mathbb{Z}} c_j^2 = 1$).

2. Basic Estimates

In this section, we shall improve the *a priori* estimate of $\|\nabla u\|_{L^2(\mathbb{R}^+;L^p)}$, which was obtained by Desjardins [15] in the case of $\mathbb{T}^2$, to be that of $\|\nabla u\|_{L^1(\mathbb{R}^+;L^p)}$ for any $p \in (1/\delta, p^*]$ with p^* being determined by (1.5). This will be one of the crucial ingredient for us to prove the $L^1(\mathbb{R}^+;\dot{B}^1_{\infty,1}(\mathbb{R}^2))$ estimate of the velocity field in Section 3. The main idea to achieve the estimate of $\|\nabla u\|_{L^1(\mathbb{R}^+;L^p)}$ is to use the decay estimate for velocity field in [18, 24, 25] and the energy method in [15].

Proposition 2.1. *Let $f(t)$ be a positive smooth function, let (ρ, u) be a smooth enough solution of* (1.1) *on $[0,T^*)$ for some positive time T^*. Then under the assumption* (1.6), *one has*

$$
\begin{aligned}
&\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx\Big) + f(t)\int_{\mathbb{R}^2}|\partial_t u|^2\,dt' \\
&\leq 4f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx + C_{m,M}\Big(f(t)(1+\|u\|_{L^2}^2)\|\nabla u\|_{L^2}^4\Big) \quad \textit{for} \quad t\in[0,T^*),
\end{aligned}
\tag{2.1}
$$

where $C_{m,M}$ is a positive constant depending on m, M in (1.6).

Proof. The proof of this proposition basically follows from that of Theorem 1 in [15]. For completeness, we outline its proof here. Indeed thanks to (1.6), one has

$$
m \leq \rho(t,x) \leq M \quad \text{for} \quad t\in[0,T^*).
\tag{2.2}
$$

In what follows, the uniform constant C always depends on m, M and sometimes on $\|\mu'\|_{L^\infty}$ also, yet we neglect the subscripts m, M for simplicity.

By taking L^2 inner product of the momentum equation of (1.1) with $\partial_t u$ and using integration by parts, we deduce from the derivation of (29) in [15] that

$$\begin{aligned}
&f(t)\int_{\mathbb{R}^2}\rho|\partial_t u|^2\,dx+\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx\Big)\\
&=f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx-f(t)\int_{\mathbb{R}^2}\partial_t u\mid\big(\rho u\cdot\nabla u\big)\,dx-f(t)\int_{\mathbb{R}^2}(u\cdot\nabla)u\mid\mathrm{div}\big(2\mu(\rho)d\big)\,dx\\
&=f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx-2f(t)\int_{\mathbb{R}^2}\partial_t u\mid\big(\rho u\cdot\nabla u\big)\,dx-f(t)\int_{\mathbb{R}^2}\rho|u\cdot\nabla u|^2\,dx\\
&\quad-f(t)\int_{\mathbb{R}^2}u\cdot\nabla u\mid\nabla\Pi\,dx,
\end{aligned}$$

where in the last step we used the momentum equation of (1.1) so that $\mathrm{div}\big(2\mu(\rho)d\big)=\rho\partial_t u+\rho u\cdot\nabla u+\nabla\Pi$. This gives rise to

$$\begin{aligned}
&f(t)\int_{\mathbb{R}^2}\rho|\partial_t u|^2\,dx+\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx\Big)\\
&\leq 2\Big(f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx+f(t)\|\sqrt{\rho}u\cdot\nabla u\|_{L^2}^2-f(t)\int_{\mathbb{R}^2}u\cdot\nabla u\mid\nabla\Pi\,dx\Big)\\
&\leq 2f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx+Cf(t)\Big(\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2+\Big|\sum_{i,k=1}^2\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\Big|\Big)
\end{aligned}\tag{2.3}$$

To deal with the pressure function Π, we get, by taking space divergence to the momentum equation of (1.1), that

$$\Pi=(-\Delta)^{-1}\mathrm{div}\big(\rho\partial_t u+\rho u\cdot\nabla u\big)-(-\Delta)^{-1}\mathrm{div}\otimes\mathrm{div}\big(2\mu(\rho)d\big),\tag{2.4}$$

from which, we deduce

$$\begin{aligned}
\Big|\sum_{i,k=1}^2\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\Big|&\lesssim\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2\\
&\quad+\|(-\Delta)^{-1}\mathrm{div}\big(\rho\partial_t u+\rho(u\cdot\nabla)u\big)\|_{BMO}\Big\|\sum_{i,k=1}^2\partial_i u^k\partial_k u^i\Big\|_{\mathscr{H}^1},
\end{aligned}$$

where $\|f\|_{\mathscr{H}^1}$ denotes the Hardy norm of f. Yet as $\mathrm{div}\,u=0$, it follows from [10] that

$$\Big\|\sum_{i,k=1}^2\partial_i u^k\partial_k u^i\Big\|_{\mathscr{H}^1}\lesssim\|\nabla u\|_{L^2}^2,$$

and $\|f\|_{BMO(\mathbb{R}^2)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^2)}$, we obtain

$$\Big|\sum_{i,k=1}^2\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\Big|\lesssim\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2+\|\rho\partial_t u+\rho(u\cdot\nabla)u\|_{L^2}\|\nabla u\|_{L^2}^2,$$

which along with $\|u\|_{L^4}^2\lesssim\|u\|_{L^2}\|\nabla u\|_{L^2}$ and (2.3) ensures that

$$\begin{aligned}
&f(t)\int_{\mathbb{R}^2}\rho|\partial_t u|^2\,dx+\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx\Big)\\
&\leq 3f'(t)\int_{\mathbb{R}^2}\mu(\rho)d:d\,dx+C\Big(f(t)\|\nabla u\|_{L^2}^4+f(t)(1+\|u\|_{L^2})\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2\Big).
\end{aligned}\tag{2.5}$$

To handle $\|\nabla u\|_{L^4}$, we write

$$\nabla u=\nabla(-\Delta)^{-1}\mathbb{P}\mathrm{div}\big(2(\mu(\rho)-1)d\big)-\nabla(-\Delta)^{-1}\mathbb{P}\mathrm{div}\big(2\mu(\rho)d\big),\tag{2.6}$$

which together with the following interpolation inequality from [9]

$$\|f\|_{L^r(\mathbb{R}^2)} \leq C\sqrt{r}\|f\|_{L^2(\mathbb{R}^2)}^{\frac{2}{r}}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{r}}, \qquad 2 \leq r < \infty, \tag{2.7}$$

ensures that for any $p \in [2, \infty)$

$$\|\nabla u\|_{L^p} \leq C_0 p\|\mu(\rho_0) - 1\|_{L^\infty}\|\nabla u\|_{L^p} + C\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\mathbb{P}\mathrm{div}\big(2\mu(\rho)d\big)\|_{L^2}^{1-\frac{2}{p}}$$

with $C_0 > 0$ being a universal constant. Taking ε_0 sufficiently small in (1.6), we obtain for $2 \leq p \leq p^* = \frac{1}{2C_0\|\mu(\rho_0)-1\|_{L^\infty}}$ that

$$\begin{aligned}\|\nabla u\|_{L^p} &\leq C\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\rho\partial_t u + \rho(u\cdot\nabla)u\|_{L^2}^{1-\frac{2}{p}}\\ &\leq C\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\big(\|\partial_t u\|_{L^2}^{1-\frac{2}{p}} + \|u\|_{L^4}^{1-\frac{2}{p}}\|\nabla u\|_{L^4}^{1-\frac{2}{p}}\big).\end{aligned} \tag{2.8}$$

In particular taking $p = 4$ in (2.8) results in

$$\|\nabla u\|_{L^4}^2 \leq C\big(\|\nabla u\|_{L^2}\|\partial_t u\|_{L^2} + \|u\|_{L^2}\|\nabla u\|_{L^2}^3\big). \tag{2.9}$$

Substituting the above inequality into (2.5), we obtain (2.1). This completes the proof of the proposition. ☐

Corollary 2.1. *Under the assumption of Proposition 2.1, we have*

$$\begin{aligned}&\|u\|_{L_t^\infty(L^2)}^2 + \|\nabla u\|_{L_t^2(L^2)}^2 \leq C\|u_0\|_{L^2}^2,\\ &\|\langle t'\rangle^{\frac{1}{2}}\nabla u\|_{L_t^\infty(L^2)}^2 + \|\langle t'\rangle^{\frac{1}{2}}\partial_t u\|_{L_t^2(L^2)}^2 \leq C\|\nabla u_0\|_{L^2}^2\exp\big(C\|u_0\|_{L^2}^4\big),\end{aligned} \tag{2.10}$$

for all $t \in [0, T^*)$ and where $\langle t\rangle \overset{\text{def}}{=} e + t$.

Proof. We first get, by using standard energy estimate to (1.1), that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|u|^2\,dx + \int_{\mathbb{R}^2}\mu(\rho)d : d\,dx = 0, \tag{2.11}$$

which implies the first inequality of (2.10).

Whereas taking $f(t') = \langle t'\rangle$ in (2.1) and integrating the resulting inequality over $[0, t]$, we obtain

$$\int_0^t\int_{\mathbb{R}^2}\langle t'\rangle|\partial_t u|^2\,dx\,dt' + \|\langle t'\rangle^{\frac{1}{2}}\nabla u\|_{L_t^\infty(L^2)}^2 \leq C\big(\|\nabla u_0\|_{L_t^\infty(L^2)}^2 + (1 + \|u_0\|_{L^2}^2)\int_0^t\langle t'\rangle\|\nabla u\|_{L^2}^4\,dt'\big),$$

Applying Gronwall's inequality and using the first inequality of (2.10) gives rise to

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^2}\langle t'\rangle|\partial_t u|^2\,dx\,dt' + \|\langle t'\rangle^{\frac{1}{2}}\nabla u\|_{L_t^\infty(L^2)}^2 \leq& C\|\nabla u_0\|_{L^2}^2\exp\big\{C(1 + \|u_0\|_{L^2}^2)\|\nabla u\|_{L_t^2(L^2)}^2\big\}\\ \leq& C\|\nabla u_0\|_{L^2}^2\exp\big(C\|u_0\|_{L^2}^4\big).\end{aligned}$$

This completes the proof of (2.10). ☐

Proposition 2.2. *With the additional assumption that $\rho_0 - 1 \in L^2(\mathbb{R}^2)$, $u_0 \in H^{-2\delta}(\mathbb{R}^2)$ for $\delta \in (0, \frac{1}{2})$, then under the assumption of Proposition 2.1, we have*

$$\|\langle t'\rangle^\delta u\|_{L^2} + \|\langle t'\rangle^{\delta_-}\nabla u\|_{L_t^2(L^2)} \leq C\sqrt{C_0}\exp\big(CC_0\big), \tag{2.12}$$

and

$$\|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}\nabla u\|_{L_t^\infty(L^2)} + \|\langle t'\rangle^{(\frac{1}{2}+\delta)_-}u_t\|_{L_t^2(L^2)} \leq C\sqrt{C_0}\exp\big(CC_0\big), \tag{2.13}$$

for any $t \in [0, T^*)$ and C_0 being determined by (1.7).

Remark 2.1. *Large time decay estimates for $\|u(t)\|_{L^2}$ and $\|\nabla u(t)\|_{L^2}$ were obtained by Gui and the authors in [3] for 3-D inhomogeneous Navier-Stokes system with constant viscosity. Gui and the second author proved the time decay estimate for $\|u(t)\|_{L^2}$ in (2.12) for 2-D inhomogeneous Navier-Sttokes system with variable density in [16]. Similar time decay estimates as (2.12) and (2.13) were obtained by Huang and Paicu in [18]. Note that for $p\in[1,2)$ and $\delta=\frac{1}{p}-\frac{1}{2}$, $L^p(\mathbb{R}^2)$ can be continuously imbedded into $H^{-2\delta}(\mathbb{R}^2)$, the decay estimates (2.12) and (2.13) are slightly general than that in [18], where the authors require the low frequency assumption for u_0 that $u_0\in L^p(\mathbb{R}^2)$ for $p\in[1,2)$. For completeness, here we shall outline the proof which basically follows from the corresponding argument in [25] for the classical 2-D Navier-Stokes system.*

According to [25] for classical Navier-Stokes system, the key ingredient used in the proof of the decay estimate for $\|u(t)\|_{L^2}$ in (2.12) is the following Lemma:

Lemma 2.1. *Under the assumption of Proposition 2.2, we have*

$$\|u(t)\|_{L^2}\leq C\sqrt{C_0}\frac{1}{\ln\langle t\rangle}\quad \textit{for any}\quad t\in[0,T^*). \tag{2.14}$$

Proof. Following the proofs of Theorem 3.1 of [18] and Lemma 4.4 of [16], we first deduce from (2.11) that

$$\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2+2\underline{\mu}\|\nabla u\|_{L^2}^2\leq 0. \tag{2.15}$$

Applying Schonbek's strategy in [24], by splitting the phase space $\mathbb{R}^2$ into two time-dependent domain: $\mathbb{R}^2=S(t)\cup S(t)^c$, where $S(t)\stackrel{\text{def}}{=}\big\{\xi:\ |\xi|\leq\sqrt{\frac{M}{2\underline{\mu}}}g(t)\ \big\}$ for some $g(t)$, which will be chosen later on. Then we deduce from (2.15) that

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2+g^2(t)\|\sqrt{\rho}u(t)\|_{L^2}^2\leq Mg^2(t)\int_{S(t)}|\hat{u}(t,\xi)|^2\,d\xi. \tag{2.16}$$

To deal with the low frequency part of u on the right hand side of (2.16), we write

$$u(t)=e^{t\Delta}u_0+\int_0^t e^{(t-t')\Delta}\mathbb{P}\Big(\mathrm{div}\big((\mu(\rho)-1)d(u)\big)+(1-\rho)(u_t+u\cdot\nabla u)-u\cdot\nabla u\Big)(s)\,dt'.$$

Taking Fourier transform with respect to x variables gives rise to

$$\begin{aligned}|\hat{u}(t,\xi)|\lesssim& e^{-t|\xi|^2}|\hat{u}_0(\xi)|+\int_0^t e^{-(t-t')|\xi|^2}\Big(|\xi|\big(\big|\mathcal{F}_x[(\mu(\rho)-1)d(u)]\big|+\big|\mathcal{F}_x(u\otimes u)\big|\big)\\&+\big|\mathcal{F}_x[(1-\rho)(u_t+u\cdot\nabla u)]\big|\Big)(t')\,dt',\end{aligned}$$

so that

$$\begin{aligned}\int_{S(t)}|\hat{u}(t,\xi)|^2\,d\xi\lesssim&\int_{S(t)}e^{-t|\xi|^2}|\hat{u}_0(\xi)|^2\,d\xi+g^4(t)\Big(\int_0^t\big(\big\|\mathcal{F}_x[(\mu(\rho)-1)d(u)]\big\|_{L^\infty_\xi}\\&+\big\|\mathcal{F}_x(u\otimes u)\big\|_{L^\infty_\xi}\big)\,dt'\Big)^2+g^2(t)\Big(\int_0^t\big\|\mathcal{F}_x[(1-\rho)(u_t+u\cdot\nabla u)]\big\|_{L^\infty_\xi}\,dt'\Big)^2.\end{aligned} \tag{2.17}$$

It is easy to observe that

$$\int_{S(t)}e^{-t|\xi|^2}|\hat{u}_0(\xi)|^2\,d\xi\leq\langle t\rangle^{-2\delta}\|u_0\|_{H^{-2\delta}}^2,$$

and

$$\begin{aligned}\Big(\int_0^t \big(\big\|\mathcal{F}_x[(\mu(\rho)-1)d(u)]\big\|_{L^\infty_\xi}+\big\|\mathcal{F}_x(u\otimes u)\big\|_{L^\infty_\xi}\big)\,dt'\Big)^2\\ \lesssim \Big(\int_0^t \big(\|(\mu(\rho)-1)\mathcal{M}(u)\|_{L^1}+\|u\otimes u\|_{L^1}\big)\,dt'\Big)^2\\ \lesssim \|\mu(\rho)-1\|^2_{L^\infty_t(L^2)}\|\nabla u\|^2_{L^2_t(L^2)}t+\Big(\int_0^t\|u(t')\|^2_{L^2}\,dt'\Big)^2,\end{aligned}$$

Finally thanks to (2.9) and (2.10), we have

$$\Big(\int_0^t \|u_t(t')\|_{L^2}\,dt'\Big)^2\leq C\ln\langle t\rangle\int_0^t\langle t'\rangle\|u_t(t')\|^2_{L^2}\,dt'\leq C\|\nabla u_0\|^2_{L^2}\exp\big(C\|u_0\|^4_{L^2}\big)\ln\langle t\rangle,$$

and

$$\begin{aligned}\int_0^t\|u\|_{L^4}\|\nabla u\|_{L^4}\,dt'\lesssim&\int_0^t\big(\|u\|^{\frac12}_{L^2}\|\nabla u\|_{L^2}\|u_t\|^{\frac12}_{L^2}+\|u\|_{L^2}\|\nabla u\|^2_{L^2}\big)\,dt'\\ \lesssim&\|u\|^{\frac12}_{L^\infty_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}\|\langle t\rangle^{\frac12}u_t\|^{\frac12}_{L^2}\ln^{\frac14}\langle t\rangle+\|u\|_{L^\infty_t(L^2)}\|\nabla u\|^2_{L^2_t(L^2)}\\ \leq& C\|\nabla u_0\|^{\frac12}_{L^2}\exp\big(C\|u_0\|^4_{L^2}\big)\ln^{\frac14}\langle t\rangle,\end{aligned}$$

which leads to

$$\begin{aligned}&\Big(\int_0^t\big\|\mathcal{F}_x[(1-\rho)(u_t+u\cdot\nabla u)]\big\|_{L^\infty_\xi}(t')\,dt'\Big)^2\\ &\leq\|(1-\rho)\|^2_{L^\infty_t(L^2)}\Big[\Big(\int_0^t\|u_t(t')\|_{L^2}\,dt'\Big)^2+\Big(\int_0^t\|u\|_{L^4}\|\nabla u\|_{L^4}\,dt'\Big)^2\Big]\\ &\leq C\|\rho_0-1\|^2_{L^2}\|\nabla u_0\|^2_{L^2}\exp\big(C\|u_0\|^4_{L^2}\big)\ln\langle t\rangle.\end{aligned}$$

Resuming the above estimates into (2.17) and then using (2.16), we obtain

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}u(t)\|^2_{L^2}+g^2(t)\|\sqrt{\rho}u(t)\|^2_{L^2}\leq& Mg^6(t)\Big(\int_0^t\|u(t')\|^2_{L^2}\,dt'\Big)^2\\ &+CC_0\Big(g^2(t)\langle t\rangle^{-2\delta}+g^6(t)\langle t\rangle+g^4(t)\ln\langle t\rangle\Big),\end{aligned}\tag{2.18}$$

for C_0 given by (1.7). Taking $g^2(t)=\frac{3}{\langle t\rangle\ln\langle t\rangle}$ in the above inequality and then integrating the resulting inequality over $[0,t]$ resulting (2.14). □

We now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. With Lemma 2.1 and (2.18), the decay estimate of $\|u(t)\|_{L^2}$ in (2.12) follows by an standard argument as [25] for the classical 2-D Navier-Stokes system (One may check page 310-311 of [25] for details). Whereas multiplying (2.15) by $\langle t\rangle^{(2\delta)-}$ and then integrating the resulting inequality over $[0,t]$, we obtain

$$\begin{aligned}\|\langle t\rangle^{\delta-}u\|^2_{L^2}+2\underline{\mu}\|\langle t'\rangle^{\delta-}\nabla u\|^2_{L^2_t(L^2)}\leq& C\big(\|u_0\|^2_{L^2}+\int_0^t\langle t'\rangle^{(2\delta-1)-}\|u(t')\|^2_{L^2}\,dt'\big)\\ \leq& CC_0\exp\big(CC_0\big),\end{aligned}\tag{2.19}$$

for C_0 given by (1.7). This proves (2.12).

On the other hand, taking $f(t)=\langle t\rangle^{(1+2\delta)-}$ in (2.1), and then using (2.19) and Gronwall's inequality, we obtain (2.13). This completes the proof of the Proposition. □

Notation: In all that follows, for C_0 given by (1.7), we already denote

$$C_1 \stackrel{\text{def}}{=} C\sqrt{C_0}\exp\bigl(CC_0\bigr). \tag{2.20}$$

We now present the key estimate in this section:

Proposition 2.3. *Under the assumptions of Proposition 2.2, for $p\in[2,p^*]$ with p^* being determined by* (1.5), *we have for any $t\in[0,T^*)$*

$$\|\langle t'\rangle^{(\frac12+\delta-\frac1p)_-}\nabla u\|_{L^2_t(L^p)}\leq \sqrt{p}C_1^{2-\frac2p}. \tag{2.21}$$

Proof. We first, get by resuming (2.9) into (2.8), that

$$\|\nabla u\|_{L^p}\leq C\sqrt{p}\Bigl(\|\nabla u\|_{L^2}^{\frac2p}\|u_t\|_{L^2}^{1-\frac2p}+\|u\|_{L^2}^{\frac12(1-\frac2p)}\|\nabla u\|_{L^2}\|u_t\|_{L^2}^{\frac12(1-\frac2p)}+\|u\|_{L^2}^{1-\frac2p}\|\nabla u\|_{L^2}^{2-\frac2p}\Bigr).$$

Notice that $p\geq 2$, multiplying $\langle t'\rangle^{(\frac12+\delta-\frac1p)_-}$ to the above inequality and then taking $L^2(0,t)$ norm of the resulting inequality, we obtain

$$\begin{aligned}\|\langle t'\rangle^{(\frac12+\delta-\frac1p)_-}\nabla u\|_{L^2_t(L^p)}\leq{}& C\sqrt{p}\Bigl(\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2_t(L^2)}^{\frac2p}\|\langle t'\rangle^{(\frac12+\delta)_-}u_t\|_{L^2_t(L^2)}^{1-\frac2p}\\&+\|\langle t'\rangle^{\delta}u\|_{L^\infty_t(L^2)}^{\frac12-\frac1p}\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|_{L^\infty_t(L^2)}^{\frac12-\frac1p}\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2_t(L^2)}^{\frac12+\frac1p}\|\langle t'\rangle^{(\frac12+\delta)_-}u_t\|_{L^2_t(L^2)}^{\frac12-\frac1p}\\&+\|u\|_{L^\infty_t(L^2)}^{1-\frac2p}\|\langle t'\rangle^{(\frac12+\delta)_-}\nabla u\|_{L^\infty_t(L^2)}^{1-\frac2p}\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2_t(L^2)}\Bigr).\end{aligned}$$

Then we get, by resuming (2.12) and (2.13) into the above inequality, that

$$\|\langle t'\rangle^{(\frac12+\delta-\frac1p)_-}\nabla u\|_{L^2_t(L^p)}\leq\sqrt{p}C_1^{2-\frac2p}\bigl(1+\|u_0\|_{L^2}^{1-\frac2p}\bigr),$$

which together with (1.7) and (2.20) leads to (2.21). □

3. The $L^1(\mathbb{R}^+;\dot B^1_{\infty,1})$ estimate for the velocity field

The goal of the this section is to present the *a priori* $L^1(\mathbb{R}^+;\dot B^1_{\infty,1})$ estimate for the velocity field, which is the most important ingredient used in the proof of Theorem 1.2.

Lemma 3.1. *Let $q\in(1/\delta,p^*]$ with p^* being determined by* (1.5) *and $\varepsilon>0$ such that $\frac2q+\varepsilon<1$. Let $(\rho,u,\nabla\Pi)$ be a smooth enough solution of* (1.1) *on $[0,T^*)$. Then under the assumptions Proposition 2.2, one has*

$$\|u\|_{\widetilde L^1_t(\dot B^{1+\frac2q+\varepsilon}_{q,\infty})}\leq\|u_0\|_{\dot B^{-1+\frac2q+\varepsilon}_{q,\infty}}+CC_1^2\bigl(1+\|\rho\|_{L^\infty_t(B^{\frac2q+\varepsilon}_{\infty,\infty})}\bigr)\quad\text{for any}\quad t<T^*, \tag{3.1}$$

where the norm $\|u\|_{\widetilde L^1_t(\dot B^{1+\frac2q+\varepsilon}_{q,\infty})}$ is given by Definition A.2 and the constant C_1 by (2.20).

Proof. Let $\mathbb{P}\stackrel{\text{def}}{=}I-\nabla(\Delta)^{-1}\mathrm{div}$ be Leray projection operator. We get, by first dividing the momentum equation of (1.1) by ρ and then applying the resulting equation by $\mathbb{P}$, that

$$\partial_t u+\mathbb{P}\bigl\{u\cdot\nabla u\bigr\}-\mathbb{P}\bigl\{\frac1\rho\bigl(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\bigr)\bigr\}=0.$$

Applying $\dot\Delta_j$ to the above equation and using a standard commutator's process yields

$$\begin{aligned}&\rho\partial_t\dot\Delta_j u+\rho u\cdot\nabla\dot\Delta_j u-\Delta\dot\Delta_j u-2\mathrm{div}\bigl((\mu(\rho)-1)\mathbb{P}d(\dot\Delta_j u)\bigr)\\&\quad=-\rho[\dot\Delta_j\mathbb{P};u\cdot\nabla]u+\rho[\dot\Delta_j\mathbb{P};\frac1\rho]\bigl(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\bigr)+2\mathrm{div}[\dot\Delta_j\mathbb{P};\mu(\rho)]d.\end{aligned} \tag{3.2}$$

Throughout this paper, we always denote $d(v)\stackrel{\text{def}}{=}\bigl(\frac12(\partial_i v_j+\partial_j v_i)\bigr)_{2\times2}$, and abbreviate $d(u)$ as d.

Taking L^2 inner product of (3.2) with $|\dot\Delta_j u|^{q-2}\dot\Delta_j u$, we obtain

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\dot\Delta_j u|^q\,dx-\int_{\mathbb{R}^2}\Delta\Delta_j u\ |\ |\dot\Delta_j u|^{q-2}\dot\Delta_j u\,dx\\
&\leq \|\dot\Delta_j u\|_{L^q}^{q-1}\Big\{C(q-1)2^j\|(\mu(\rho)-1)\mathbb{P}d(\dot\Delta_j u)\|_{L^q}+\|\rho[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^q}\\
&\quad+\|\rho[\dot\Delta_j\mathbb{P};\frac{1}{\rho}]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^q}+c(q-1)2^j\|[\dot\Delta_j\mathbb{P},\mu(\rho)](2d)\|_{L^q}\Big\}.
\end{aligned}\tag{3.3}$$

However as $\mathrm{div}\,u=0$, one gets, by using integration by parts and Lemma A.5 of [12], that

$$\begin{aligned}
-\int_{\mathbb{R}^2}\Delta\dot\Delta_j u\ |\ |\dot\Delta_j u|^{q-2}\dot\Delta_j u\,dx=&\int_{\mathbb{R}^2}|\dot\Delta_j\nabla u|^2|\dot\Delta_j u|^{q-2}\,dx\\
&+(q-2)\int_{\mathbb{R}^2}|\dot\Delta_j u|^{q-2}\big(\nabla|\dot\Delta_j u|\big)^2\,dx\geq c2^{2j}\|\dot\Delta_j u\|_{L^q}^q,
\end{aligned}$$

for some positive constant c.

Whereas it follows from Lemma A.1 that

$$\|(\mu(\rho)-1)\mathbb{P}d(\dot\Delta_j u)\|_{L^q}\lesssim 2^j\|\mu(\rho)-1\|_{L^\infty}\|\dot\Delta_j u\|_{L^q}\lesssim 2^j\|\mu(\rho_0)-1\|_{L^\infty}\|\dot\Delta_j u\|_{L^q}.$$

Therefore taking ε_0 sufficiently small in (1.6) and using (2.2), we deduce from (3.3) that

$$\begin{aligned}
\frac{d}{dt}\|\rho^{\frac1q}\dot\Delta_j u\|_{L^q}+c2^{2j}\|\rho^{\frac1q}\dot\Delta_j u\|_{L^q}\lesssim&\|\rho[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^q}\\
&+\|\rho[\dot\Delta_j\mathbb{P};\frac{1}{\rho}]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^q}+2^j\|[\dot\Delta_j\mathbb{P},\mu(\rho)](2d)\|_{L^q},
\end{aligned}$$

which gives rise to

$$\begin{aligned}
\|\rho^{\frac1q}\dot\Delta_j u(t)\|_{L^q}\lesssim&e^{-c2^{2j}t}\|\rho_0^{\frac1q}\dot\Delta_j u_0\|_{L^q}+\int_0^t e^{-c2^{2j}(t-t')}\Big\{\|\rho[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^q}\\
&+\|\rho[\dot\Delta_j\mathbb{P};\frac{1}{\rho}]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^q}+2^j\|[\dot\Delta_j\mathbb{P},\mu(\rho)](2d)\|_{L^q}\Big\}\,dt'.
\end{aligned}$$

As a consequence, thanks to (2.2) and Definition A.2, we conclude, for $q\in(2,p^*]$, that

$$\begin{aligned}
\|u\|_{\widetilde L^1_t(\dot B^{1+\frac2q+\varepsilon}_{q,\infty})}\lesssim&\|u_0\|_{\dot B^{-1+\frac2q+\varepsilon}_{q,\infty}}+\sup_j 2^{(-1+\frac2q+\varepsilon)j}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1_t(L^q)}\\
&+\sup_j 2^{(-1+\frac2q+\varepsilon)j}\|[\dot\Delta_j\mathbb{P};\frac1\rho]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^1_t(L^q)}\\
&+\sup_j 2^{(\frac2q+\varepsilon)j}\|[\dot\Delta_j\mathbb{P};\mu(\rho)]d\|_{L^1_t(L^q)}.
\end{aligned}\tag{3.4}$$

In what follows, we shall handle term by term the right-hand side of (3.4). Firstly appplying Bony's decomposition (A.5), one has

$$[\dot\Delta_j\mathbb{P};u\cdot\nabla]u=[\dot\Delta_j\mathbb{P};T_u\cdot\nabla]u+\dot\Delta_j\mathbb{P}R(u,\nabla u)-R(u,\nabla\dot\Delta_j u).$$

Applying Lemma 1 of [23] gives

$$\begin{aligned}
2^{(-1+\frac2q+\varepsilon)j}\|[\dot\Delta_j\mathbb{P};T_u\cdot\nabla]u\|_{L^q}\lesssim&2^{(-1+\frac2q+\varepsilon)j}\sum_{|j-\ell|\leq4}\|\nabla\dot S_{\ell-1}u\|_{L^\infty}\|\dot\Delta_\ell u\|_{L^q}\\
&\lesssim\|\nabla u\|_{L^q}\|u\|_{\dot H^{\frac2q+\varepsilon}}\lesssim\|\nabla u\|_{L^q}\|\nabla u\|_{L^2}^{\frac2q+\varepsilon}\|u\|_{L^2}^{1-\frac2q-\varepsilon}.
\end{aligned}$$

Whereas applying Lemma A.1, one has

$$2^{(-1+\frac2q+\varepsilon)j}\|\dot\Delta_j\mathbb{P}R(u,\nabla u)\|_{L^q}\lesssim 2^{(\frac2q+\varepsilon)j}\sum_{\ell\geq j-3}\|\dot\Delta_\ell u\|_{L^2}\|\dot S_{\ell+2}\nabla u\|_{L^q}\lesssim\|u\|_{\dot H^{\frac2q+\varepsilon}}\|\nabla u\|_{L^q}.$$

The same estimate holds for $R(u,\nabla\dot\Delta_j u)$. which together with (2.10) and Proposition 2.3 implies

$$\begin{aligned}\sup_j 2^{(-1+\frac{2}{q}+\varepsilon)j}\|[\dot\Delta_j\mathbb{P},u\cdot\nabla]u\|_{L^1_t(L^q)}&\leq C\|u\|_{L^\infty_t(L^2)}^{1-\frac{2}{q}-\varepsilon}\|\nabla u\|_{L^\infty_t(L^2)}^{\frac{2}{q}+\varepsilon}\|\nabla u\|_{L^1_t(L^q)}\\ &\leq C\big(\|u_0\|_{L^2}+\|\nabla u_0\|_{L^2}\exp\big(C\|u_0\|_{L^2}^4\big)\big)\|\langle t'\rangle^{(\frac12+\delta-\frac1q)_-}\nabla u\|_{L^2_t(L^q)}\leq CC_1^2,\end{aligned}\tag{3.5}$$

where in the last step, we used the assumption that $q\delta>1$ so that $\|\langle t'\rangle^{-(\frac12+\delta-\frac1q)_-}\|_{L^2_t}\leq C.$

Exactly along the same line to the proof of (3.5), we get, by applying Bony's decomposition (A.5), that

$$[\dot\Delta_j\mathbb{P},\frac{1}{\rho}]f=[\dot\Delta_j\mathbb{P},T_{\frac{1}{\rho}}]f+\dot\Delta_j\mathbb{P}R(\frac{1}{\rho},f)-R(\frac{1}{\rho},\dot\Delta_j\mathbb{P}f).$$

It follows from Lemma 1 of [23] that

$$\begin{aligned}2^{(-1+\frac{2}{q}+\varepsilon)j}\|[\dot\Delta_j\mathbb{P},T_{\frac{1}{\rho}}]f\|_{L^q}&\lesssim 2^{j\varepsilon}\|[\dot\Delta_j\mathbb{P},T_{\frac{1}{\rho}}]f\|_{L^2}\\ &\lesssim\|\nabla\frac{1}{\rho}\|_{\dot B^{-1+\varepsilon}_{\infty,\infty}}\|f\|_{L^2}\lesssim(1+\|\rho\|_{L^\infty})\|\rho\|_{B^\varepsilon_{\infty,\infty}}\|f\|_{L^2},\end{aligned}$$

and applying Lemma A.1 leads to

$$\begin{aligned}2^{(-1+\frac{2}{q}+\varepsilon)j}\|\dot\Delta_j\mathbb{P}R(\frac{1}{\rho},f)\|_{L^q}&\lesssim 2^{j\varepsilon}\|\dot\Delta_j\mathbb{P}R(\frac{1}{\rho},f)\|_{L^2}\\ &\lesssim 2^{j\varepsilon}\sum_{\ell\geq j-3}\|\dot\Delta_\ell(\frac{1}{\rho})\|_{L^\infty}\|\dot S_{\ell+2}f\|_{L^2}\lesssim(1+\|\rho\|_{L^\infty})\|\rho\|_{B^\varepsilon_{\infty,\infty}}\|f\|_{L^2}.\end{aligned}$$

The same estimate holds for $R(\frac{1}{\rho},\dot\Delta_j\mathbb{P}f)$, so we obtain

$$2^{(-1+\frac{2}{q}+\varepsilon)j}\big\|[\dot\Delta_j\mathbb{P},\frac{1}{\rho}]f\big\|_{L^1_t(L^q)}\leq C\|\rho\|_{L^\infty_t(B^\varepsilon_{\infty,\infty})}\|f\|_{L^1_t(L^2)},$$

from which and $\mathrm{div}(2\mu(\rho)d)-\nabla\Pi=\rho\partial_t u+\rho(u\cdot\nabla)u$, we deduce that

$$\begin{aligned}\sup_j 2^{(-1+\frac{2}{q}+\varepsilon)j}&\|[\dot\Delta_j\mathbb{P},\frac{1}{\rho}]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^1_t(L^q)}\\ &\leq C\|\rho\|_{L^\infty_t(B^\varepsilon_{\infty,\infty})}\|(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi)\|_{L^1_t(L^2)}\\ &\leq C\|\rho\|_{L^\infty_t(B^\varepsilon_{\infty,\infty})}\big(\|\partial_t u\|_{L^1_t(L^2)}+\int_0^t\|u\|_{L^4}\|\nabla u\|_{L^4}\,dt'\big).\end{aligned}$$

However, notice from (2.9) and Proposition 2.2 that

$$\begin{aligned}\int_0^t\|u\|_{L^4}\|\nabla u\|_{L^4}\,dt'\leq&C\int_0^t\big(\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}\|u_t\|_{L^2}^{\frac12}+\|u\|_{L^2}\|\nabla u\|_{L^2}^2\big)\,dt'\\ \lesssim&\|u\|_{L^\infty_t(L^2)}^{\frac12}\|\nabla u\|_{L^2_t(L^2)}\|\langle t'\rangle^{(\frac12+\delta)_-}u_t\|_{L^2_t(L^2)}^{\frac12}\|\langle t'\rangle^{-(\frac14+\frac{\delta}{2})_-}\|_{L^4_t}\\ &+\|u\|_{L^\infty_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}^2\leq C\big(C_1^{\frac12}\|u_0\|_{L^2}^{\frac32}+\|u_0\|_{L^2}^3\big),\end{aligned}\tag{3.6}$$

and

$$\|\partial_t u\|_{L^1_t(L^2)}\leq C\|\langle t'\rangle^{(\frac12+\delta)_-}\partial_t u\|_{L^2_t(L^2)}\leq C_1,$$

so that we obtain

$$\sup_j 2^{(-1+\frac{2}{q}+\varepsilon)j}\|[\dot\Delta_j\mathbb{P},\frac{1}{\rho}]\big(\mathrm{div}(2\mu(\rho)d)-\nabla\Pi\big)\|_{L^1_t(L^q)}\leq CC_1\|\rho\|_{L^\infty_t(B^\varepsilon_{\infty,\infty})}.\tag{3.7}$$

As $\frac{2}{q}+\varepsilon<1$, the same process also ensures

$$\begin{aligned}\sup_j 2^{(\frac{2}{q}+\varepsilon)j}\|[\dot\Delta_j\mathbb{P},\mu(\rho)](2d)\|_{L^1_t(L^q)} &\leq C\|\rho\|_{L^\infty_t(B^{\frac{2}{q}+\varepsilon}_{\infty,\infty})}\|\nabla u\|_{L^1_t(L^q)}\\ &\leq C\|\rho\|_{L^\infty_t(B^{\frac{2}{q}+\varepsilon}_{\infty,\infty})}\|\langle t'\rangle^{(\frac12+\delta-\frac1q)_-}\nabla u\|_{L^2_t(L^q)}\leq CC_1^2\|\rho\|_{L^\infty_t(B^{\frac{2}{q}+\varepsilon}_{\infty,\infty})}.\end{aligned}\tag{3.8}$$

Substituting (3.5), (3.7) and (3.8) into (3.4) results in (3.1), and we complete the proof of Lemma 3.1. □

With Lemma 3.1, we can prove the *a priori* $L^1(\mathbb{R}^+;\,\dot B^1_{\infty,1}(\mathbb{R}^2))$ estimate for u.

Proposition 3.1. *Under the assumptions of Lemma 3.1, there exists a positive constant C which depends on m, M and $\|\mu'\|_{L^\infty}$ such that if*

$$4C^2C_1^2\big(1+\|\rho_0\|_{B^{\frac{2}{q}+\varepsilon}_{\infty,\infty}}\big)\|\mu(\rho_0)-1\|_{L^\infty}\leq 1,\tag{3.9}$$

for C_1 given by (2.20), one has

$$\|u\|_{L^1_t(\dot B^1_{\infty,1})}\leq 2CC_1^2\big(1+\|\rho_0\|_{B^{\frac{2}{q}+\varepsilon}_{\infty,\infty}}\big).\tag{3.10}$$

Proof. Bony's decomposition (A.5) for $(\mu(\rho)-1)d$ reads

$$(\mu(\rho)-1)d=T_{\mu(\rho)-1}d+T_d(\mu(\rho)-1)+\mathcal{R}(\mu(\rho)-1,d).$$

Applying para-product estimates ([6]) gives

$$\begin{aligned}\|T_{\mu(\rho)-1}d\|_{L^1_t(\dot B^0_{\infty,1})}&\lesssim\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot B^1_{\infty,1})}\\ &\lesssim\|\mu(\rho_0)-1\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})}.\end{aligned}\tag{3.11}$$

To deal with $\mathcal{R}(\mu(\rho)-1,d)$, for any integer N, we decompose it as

$$\begin{aligned}\|\mathcal{R}(\mu(\rho)-1,d)\|_{L^1_t(\dot B^0_{\infty,1})}\lesssim&\sum_{\ell\leq 0}\|\dot\Delta_\ell(\mathcal{R}(\mu(\rho)-1,d))\|_{L^1_t(L^\infty)}\\ &+\sum_{0\leq\ell\leq N}\|\dot\Delta_\ell(\mathcal{R}(\mu(\rho)-1,d))\|_{L^1_t(L^\infty)}\\ &+\sum_{N\leq\ell}\|\dot\Delta_\ell(\mathcal{R}(\mu(\rho)-1,d))\|_{L^1_t(L^\infty)}\overset{\text{def}}{=}\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

Let q be as in Lemma 3.1 and $\bar q=\frac{2q}{2+q}$. Then by virtue of Lemma A.1 and para-product estimates ([6]), we have

$$\begin{aligned}\mathrm{I}\lesssim&\|\mathcal{R}(\mu(\rho)-1,d)\|_{L^1_t(\dot B^0_{\bar q,\infty})}\\ \lesssim&\|(\mu(\rho)-1)d\|_{L^1_t(L^{\bar q})}+\|T_{\mu(\rho)-1}d\|_{L^1_t(\dot B^0_{\bar q,\infty})}+\|T_d(\mu(\rho)-1)\|_{L^1_t(\dot B^0_{\bar q,\infty})}\\ \lesssim&\|\mu(\rho)-1\|_{L^\infty_t(L^2)}\|\nabla u\|_{L^1_t(L^q)}\lesssim C_1^2\|\rho_0-1\|_{L^2},\end{aligned}$$

where in the last step, we used (2.21). Along the same line, one has

$$\mathrm{II}\lesssim N\|\mathcal{R}(\mu(\rho)-1,d)\|_{L^1_t(\dot B^0_{\infty,\infty})}\lesssim N\|\mu(\rho_0)-1\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})},$$

and

$$\begin{aligned}\mathrm{III}\lesssim&\sum_{\ell\geq N}\sum_{j\geq\ell-N_0}\|\dot\Delta_j(\mu(\rho)-1)\|_{L^\infty_t(L^\infty)}\|\dot\Delta_j(\nabla u)\|_{L^1_t(L^\infty)}\\ \lesssim&\sum_{\ell\geq N}2^{-\ell\varepsilon}\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}\|u\|_{\widetilde L^1_t(\dot B^{1+\varepsilon}_{\infty,\infty})}\\ \lesssim&2^{-N\varepsilon}\|\mu(\rho_0)-1\|_{L^\infty}\|u\|_{\widetilde L^1_t(\dot B^{1+\frac{2}{q}+\varepsilon}_{q,\infty})}\end{aligned}$$

for any $q\in(2,p^*)$ with $q\varepsilon>1$. Hence we obtain

$$\begin{aligned}\|\mathcal{R}(\mu(\rho)-1,d)\|_{L^1_t(\dot B^0_{\infty,1})}\leq & CC_1^2+C\|\mu(\rho_0)-1\|_{L^\infty}\\ &\times\Big(N\|u\|_{L^1_t(\dot B^1_{\infty,1})}+2^{-N\varepsilon}\|u\|_{L^1_t(\dot B^{1+\frac2q+\varepsilon}_{q,\infty})}\Big).\end{aligned}\tag{3.12}$$

The same process leads to

$$\begin{aligned}\|T_d(\mu(\rho)-1)\|_{L^1_t(\dot B^0_{\infty,1})}\leq CC_1^2+C\Big(&\|\mu(\rho_0)-1\|_{L^\infty}\|\nabla u\|_{L^1_t(L^\infty)}N\\ &+2^{-N\varepsilon}\|\mu(\rho)-1\|_{L^\infty_t(\dot B^\varepsilon_{\infty,\infty})}\|\nabla u\|_{L^1_t(L^\infty)}\Big).\end{aligned}\tag{3.13}$$

Notice that

$$\begin{aligned}&\|u_0\|_{\dot B^{-1+\frac2q+\varepsilon}_{q,\infty}}\lesssim\|u_0\|_{\dot B^\varepsilon_{2,\infty}}\lesssim\|u_0\|_{H^1},\\ &\|\rho\|_{L^\infty_t(B^{\frac2q+\varepsilon}_{\infty,\infty})}\leq\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big),\\ &\|\mu(\rho)-1\|_{L^\infty_t(B^\varepsilon_{\infty,\infty})}\leq\|\mu(\rho_0)-1\|_{B^\varepsilon_{\infty,\infty}}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big),\end{aligned}$$

and Riesz transform maps continuously from $\dot B^0_{\infty,1}$ from $\dot B^0_{\infty,1}$ with uniform bound, we get, by summing up (3.11), (3.12), (3.13) and Lemma 3.1, that

$$\begin{aligned}\|\mathcal{R}\mathbb{P}\mathcal{R}\cdot\big(2(\mu(\rho)-1)d\big)\|_{L^1_t(\dot B^0_{\infty,1})}\leq & CC_1^2+C\Big(\|\mu(\rho_0)-1\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})}N\\ &+C_1^22^{-N\varepsilon}\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big)\big)\\ &+2^{-N\varepsilon}\|\mu(\rho_0)-1\|_{B^\varepsilon_{\infty,\infty}}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big)\Big).\end{aligned}$$

Let $[Y]$ be the integer part of Y. Then choosing $N=\Big[\frac{C}{\varepsilon\ln 2}\|\nabla u\|_{L^1_t(L^\infty)}\Big]$ so that

$$C2^{-N\varepsilon}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big)\leq 1$$

in the above inequality results in

$$\|\mathcal{R}\mathbb{P}\mathcal{R}\cdot\big(2(\mu(\rho)-1)d\big)\|_{L^1_t(\dot B^0_{\infty,1})}\leq CC_1^2\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\big)+C\|\mu(\rho_0)-1\|_{L^\infty}\|u\|^2_{L^1_t(\dot B^1_{\infty,1})}.\tag{3.14}$$

To handle $\mathcal{R}(-\Delta)^{-\frac12}\mathbb{P}\mathrm{div}\big(2\mu(\rho)d\big)$, for any integer L, we get, by applying Lemma A.1, that

$$\begin{aligned}&\|\mathcal{R}(-\Delta)^{-\frac12}\mathbb{P}\mathrm{div}\big(2\mu(\rho)d\big)\|_{L^1_t(\dot B^0_{\infty,1})}\\ &\lesssim\sum_{\ell\leq0}2^{\frac{2\ell}{q}}\|\mu(\rho)d\|_{L^1_t(L^q)}+\sum_{0\leq\ell\leq L}\|\dot\Delta_\ell(\rho\partial_tu+\rho(u\cdot\nabla)u)\|_{L^1_t(L^2)}\\ &\quad+\sum_{L\leq\ell}2^{-\ell\varepsilon}\|\mu(\rho)d\|_{\widetilde L^1_t(\dot B^\varepsilon_{\infty,\infty})}\\ &\lesssim\|\nabla u\|_{L^1_t(L^q)}+(\|\partial_tu\|_{L^1_t(L^2)}+\|u\cdot\nabla u\|_{L^1_t(L^2)})\sqrt{L}\\ &\quad+2^{-L\varepsilon}\big(\|\mu(\rho)\|_{L^\infty_t(L^\infty)}\|u\|_{\widetilde L^1_t(\dot B^{1+\frac2q+\varepsilon}_{q,\infty})}+\|\mu(\rho)\|_{\widetilde L^\infty_t(\dot B^\varepsilon_{\infty,\infty})}\|\nabla u\|_{L^1_t(L^\infty)}\big),\end{aligned}$$

from which, Lemma 3.1, (3.6) and Proposition 2.3, we infer

$$\begin{aligned}&\|\mathcal{R}(-\Delta)^{-\frac12}\mathbb{P}\mathrm{div}\big(2\mu(\rho)d\big)\|_{L^1_t(\dot B^0_{\infty,1})}\\ &\quad\leq C\Big\{C_1^2+C_1\sqrt{L}+C_1^22^{-L\varepsilon}\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\big)\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big)\Big\}.\end{aligned}$$

Taking $L=\Big[\frac{C}{\varepsilon\ln 2}\|\nabla u\|_{L^1_t(L^\infty)}\Big]$ in the above inequality results in

$$\|\mathcal{R}_i(-\Delta)^{-\frac12}\mathbb{P}_j\mathrm{div}\big(2\mu(\rho)d\big)\|_{L^1_t(\dot B^0_{\infty,1})}\leq CC_1^2\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\big)+\frac12\|\nabla u\|_{L^1_t(L^\infty)}. \tag{3.15}$$

Thus thanks to (2.6), we get, by combining (3.14) with (3.15), that

$$\|u\|_{L^1_t(\dot B^1_{\infty,1})}\leq CC_1^2\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\big)+C\|\mu(\rho_0)-1\|_{L^\infty}\|u\|^2_{L^1_t(\dot B^1_{\infty,1})}+\frac12\|\nabla u\|_{L^1_t(L^\infty)},$$

which ensures that

$$\|u\|_{L^1_t(\dot B^1_{\infty,1})}\leq CC_1^2\big(1+\|\rho_0\|_{B^{\frac2q+\varepsilon}_{\infty,\infty}}\big)+C\|\mu(\rho_0)-1\|_{L^\infty}\|u\|^2_{L^1_t(\dot B^1_{\infty,1})},$$

from which and (3.9), we conclude (3.10). This completes the proof of Proposition 3.1. □

4. The blow-up criterion of (1.1)

The purpose of this section is to prove a blow-up criterion for smooth enough solutions of (1.1). As a matter of fact, we shall prove a more general result concerning the propagation of regularities for (1.1) which does not require any smallness assumption on the fluctuation of the viscous coefficient. Toward this, let $a\stackrel{\mathrm{def}}{=}\frac1\rho-1$ and $\widetilde\mu(a)\stackrel{\mathrm{def}}{=}\mu(\frac{1}{1+a})$, we write (1.1) as:

$$\begin{cases}\partial_t a+(u\cdot\nabla)a=0\\ \partial_t u+u\cdot\nabla u+(1+a)\big\{\nabla\Pi-\mathrm{div}\big(\widetilde\mu(a)2d\big)\big\}=0\\ \mathrm{div}\,u=0\\ (a,u)_{|t=0}=(a_0,u_0).\end{cases} \tag{4.1}$$

The main result can be listed as follows, which is a similar version of blow-up criterion for hyperbolic systems ([21]).

Theorem 4.1. *Let $s>1$ and $a_0\in H^{1+s}(\mathbb{R}^2)$ satisfy*

$$0<\mathfrak{m}\leq 1+a_0\leq\mathfrak{M}. \tag{4.2}$$

Let $u_0\in H^s(\mathbb{R}^2)$ be a solenoidal vector field. Then there exists a positive time T^*, so that (4.1) has a unique solution (a,u) with $a\in\mathcal{C}([0,T];H^{1+s}(\mathbb{R}^2))$, $u\in\mathcal{C}([0,T];H^s(\mathbb{R}^2))\cap\widetilde L^1_T(H^{s+2})$ for any $T<T^*$. Moreover, if T^* is the maximum time of existence and $T^*<\infty$, there holds

$$\int_0^{T^*}\|\nabla u\|_{L^\infty}\,dt'=\infty. \tag{4.3}$$

Proof. We first deduce from the standard well-posedness theory (see [2, 11] for instance) that (4.1) has a unique solution on $[0,T^*)$ for some positive time $T^*<\infty$. Moreover, there holds

$$\mathfrak{m}\leq 1+a(t,x)\leq\mathfrak{M},\quad\text{and}\quad\|a(t)\|_{L^p}=\|a_0\|_{L^p}\quad\forall\,p\in[2,\infty],\quad t<T^*. \tag{4.4}$$

And it follows from the proof of (2.10) that

$$\|u\|^2_{L^\infty_t(L^2)}+\|\nabla u\|^2_{L^2_t(L^2)}\leq C\|u_0\|^2_{L^2}\quad\text{for}\quad t<T^*. \tag{4.5}$$

While we get, by applying $\dot\Delta_j$ to the continuous equation of (4.1) and then taking the L^2 inner product of the resulting equation with $\dot\Delta_j a$, that for all $r>0$,

$$\frac12\frac{d}{dt}\|\dot\Delta_j a\|^2_{L^2}\leq\Big|\int_{\mathbb{R}^2}[\dot\Delta_j;u]\cdot\nabla a\mid\dot\Delta_j a\,dx\Big|,$$

applying Lemma A.2 gives

$$\frac12\frac{d}{dt}\|\dot\Delta_j a\|^2_{L^2}\lesssim c_j^2(t)2^{-2jr}\big(\|\nabla u\|_{L^\infty}\|\rho-1\|_{\dot H^r}+\|\nabla\rho\|_{L^\infty}\|u\|_{\dot H^r}\big)\|a\|_{\dot H^r},$$

from which, we infer

$$\|a\|_{\widetilde{L}^\infty_t(\dot{H}^r)} \leq \|a_0\|_{\dot{H}^r} + C\int_0^t \big(\|\nabla u\|_{L^\infty}\|a\|_{\dot{H}^r} + \|\nabla a\|_{L^\infty}\|u\|_{\dot{H}^r}\big)\,dt'.$$

Applying Gronwall's inequality, (4.4), and the fact that

$$\|\nabla a\|_{L^\infty_t(L^p)} \leq \|\nabla a_0\|_{L^p}\exp\big(\|\nabla u\|_{L^1_t(L^\infty)}\big) \quad for \quad \forall\, p\in[1,\infty], \tag{4.6}$$

leads to

$$\|a\|_{\widetilde{L}^\infty_t(H^r)} \leq \big(\|a_0\|_{H^r} + C\|\nabla a_0\|_{L^\infty}\|u\|_{L^1_t(H^r)}\big)\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big). \tag{4.7}$$

On the other hand, applying Δ_q to the momentum equation of (4.1) and using Bony's decomposition (A.5) in the inhomogeneous context, one has

$$\begin{aligned}&\partial_t\Delta_q u + u\cdot\nabla\Delta_q u + \Delta_q\nabla\Pi + \Delta_q\nabla T_a\Pi - \mathrm{div}\big((1+a)\widetilde{\mu}(a)\Delta_q(2d)\big)\\&\quad= \big[\Delta_q; u\cdot\nabla\big]u + \Delta_q T_{\nabla a}\Pi - \Delta_q\mathcal{R}(a,\nabla\Pi) + R_j,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}R_q &= \Delta_q\big[(1+a)\mathrm{div}(\widetilde{\mu}(a)2d)\big] - \mathrm{div}\big((1+a)\widetilde{\mu}(a)\Delta_q(2d)\big)\\&= \Delta_q\big[a\,\mathrm{div}\big((\widetilde{\mu}(a)-1)2d\big)\big] - \mathrm{div}\big[a\,\Delta_q\big((\widetilde{\mu}(a)-1)2d\big)\big]\\&\quad + \Delta_q\big(a\,\mathrm{div}(2d)\big) - \mathrm{div}\big(a\,\Delta_q(2d)\big) - \mathrm{div}\big\{(1+a)[\Delta_q;\widetilde{\mu}(a)-1]\cdot(2d)\big\}\\&\overset{\mathrm{def}}{=} R_q^1 + \mathrm{div}\,R_q^2,\end{aligned} \tag{4.9}$$

and $R_j^2 \overset{\mathrm{def}}{=} -(1+a)[\Delta_q;\widetilde{\mu}(a)-1]\cdot(2d)$, $R_q^1 \overset{\mathrm{def}}{=} R_q - \mathrm{div}\,R_q^2$.

Taking L^2 inner product of (4.8) with $\Delta_q u$ and using $\mathrm{div}\,u=0$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \big((1+a)\widetilde{\mu}(a)\Delta_q(2d)\mid\Delta_q(2d)\big)_{L^2}\\&\quad= \Big(\big[\Delta_q; u\cdot\nabla\big]u + \Delta_q T_{\nabla a}\Pi - \Delta_q\mathcal{R}(a,\nabla\Pi) + R_q \mid \Delta_q u\Big)_{L^2}.\end{aligned}$$

Notice that $\mathfrak{m}\leq(1+a)$ and $0<\underline{\mu}\leq\mu$, then we get, by applying standard process (like [12]) and Lemma A.1, that

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(H^s)} + \|u\|_{\widetilde{L}^1_t(H^{s+2})} \lesssim &\|u_0\|_{H^s} + \|\Delta_{-1}u\|_{L^1_t(L^2)} + \Big(\sum_{q\geq-1}2^{2qs}\big\|[u;\Delta_q]\cdot\nabla u\big\|_{L^1_t(L^2)}^2\Big)^{\frac12}\\&+ \|T_{\nabla a}\Pi\|_{\widetilde{L}^1_t(\dot{H}^s)} + \|\mathcal{R}(a,\nabla\Pi)\|_{\widetilde{L}^1_t(\dot{H}^s)}\\&+ \Big(\sum_{q\geq-1}2^{2qs}\|R_q^1\|_{L^1_t(L^2)}^2\Big)^{\frac12} + \Big(\sum_{q\geq-1}2^{2q(s+1)}\|R_q^2\|_{L^1_t(L^2)}^2\Big)^{\frac12},\end{aligned} \tag{4.10}$$

where R_q^1, R_q^2 are given by (4.9). For $s>0$, applying Lemma A.2 yields

$$\big\|[u;\Delta_q]\cdot\nabla u\big\|_{L^2} \lesssim c_q(t)2^{-qs}\|\nabla u\|_{L^\infty}\|u\|_{H^s},$$

from which, we deduce

$$\begin{aligned}\Big(\sum_{q\geq-1}2^{2qs}\big\|[u;\Delta_q]\cdot\nabla u\big\|_{L^1_T(L^2)}^2\Big)^{\frac12} &\lesssim \int_0^t\Big(\sum_{q\geq-1}2^{2js}\big\|[u;\Delta_q]\cdot\nabla u\big\|_{L^2}^2\Big)^{\frac12}dt'\\&\lesssim \int_0^t\|\nabla u\|_{L^\infty}\|u\|_{H^s}\,dt'.\end{aligned} \tag{4.11}$$

Along the same line, we have

$$
\begin{aligned}
&\|T_{\nabla a}\Pi\|_{\widetilde{L}^1_t(\dot H^s)} \lesssim \int_0^t \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{\dot H^{s-1}}\,dt' \quad \text{and}\\
&\|\mathcal{R}(a,\nabla\Pi)\|_{\widetilde{L}^1_t(\dot H^s)} \lesssim \int_0^t \|a\|_{\dot B^s_{\infty,2}}\|\nabla\Pi\|_{L^2}\,dt' \lesssim \int_0^t \|a\|_{\dot H^{s+1}}\|\nabla\Pi\|_{L^2}\,dt'.
\end{aligned}
\tag{4.12}
$$

Notice that

$$
\begin{aligned}
&\Delta_q\big[a\,\mathrm{div}\big((\widetilde\mu(a)-1)2d\big)\big]-\mathrm{div}\big[a\,\Delta_q\big((\widetilde\mu(a)-1)2d\big)\big]\\
&=[\Delta_q;a]\mathrm{div}\big((\widetilde\mu(a)-1)2d\big)+[a;\mathrm{div}]\Delta_q\big((\widetilde\mu(a)-1)2d\big),
\end{aligned}
$$

applying Lemma A.2 yields

$$
\begin{aligned}
&\big\|\Delta_q\big[a\,\mathrm{div}\big((\widetilde\mu(a)-1)2d\big)\big]-\mathrm{div}\big[a\,\Delta_q\big((\widetilde\mu(a)-1)2d\big)\big]\big\|\\
&\lesssim c_q(t)2^{-qs}\big(\|\nabla a\|_{L^\infty}\|(\widetilde\mu(a)-1)2d\|_{H^s}+\|(\widetilde\mu(a)-1)2d\|_{L^\infty}\|a\|_{H^{s+1}}\big),
\end{aligned}
$$

from which, we deduce, by a similar proof of (4.11), that

$$
\begin{aligned}
&\Big(\sum_{q\geq -1}2^{2qs}\big\|\Delta_q\big[a\,\mathrm{div}\big((\widetilde\mu(a)-\mu)2d\big)\big]-\mathrm{div}\big[a\,\Delta_q\big((\widetilde\mu(a)-\mu)2d\big)\big]\big\|^2_{L^1_t(L^2)}\Big)^{\frac12}\\
&\lesssim \int_0^t\big(\|\nabla a\|_{L^\infty}\|(\widetilde\mu(a)-1)2d\|_{H^s}+\|(\widetilde\mu(a)-\mu)2d\|_{L^\infty}\|a\|_{H^{s+1}}\big)\,dt'\\
&\lesssim \int_0^t\big(\|\nabla a\|_{L^\infty}(\|\nabla u\|_{L^\infty}\|a\|_{H^s}+\|\widetilde\mu(a)-1\|_{L^\infty}\|u\|_{H^{s+1}})+\|\nabla u\|_{L^\infty}\|a\|_{H^{s+1}}\big)\,dt'.
\end{aligned}
\tag{4.13}
$$

While notice that

$$
\Delta_q\big(a\,\mathrm{div}(2d)\big)-\mathrm{div}\big(a\,\Delta_q(2d)\big)=[\Delta_q;a]\mathrm{div}(2d)+[a;\mathrm{div}]\Delta_j(2d),
$$

a similar proof of (4.13) leads to

$$
\begin{aligned}
&\Big(\sum_{q\geq -1}2^{2qs}\big\|\Delta_q\big(a\,\mathrm{div}(2d)\big)-\mathrm{div}\big(a\,\Delta_q(2d)\big)\big\|^2_{L^1_t(L^2)}\Big)^{\frac12}\\
&\qquad\qquad\lesssim \int_0^t\big(\|\nabla a\|_{L^\infty}\|u\|_{H^{s+1}}+\|\nabla u\|_{L^\infty}\|a\|_{H^{s+1}}\big)\,dt'.
\end{aligned}
$$

Therefore thanks to (4.9), we obtain

$$
\begin{aligned}
\Big(\sum_{q\geq -1}2^{2qs}\|R^1_j\|^2_{L^1_t(L^2)}\Big)^{\frac12}\lesssim&\int_0^t\big(\|\nabla a\|_{L^\infty}(\|\nabla u\|_{L^\infty}\|a\|_{H^s}\\
&+(1+\|a\|_{L^\infty})\|u\|_{H^{s+1}})+\|\nabla u\|_{L^\infty}\|a\|_{H^{s+1}}\big)\,dt'.
\end{aligned}
\tag{4.14}
$$

It follows the same line that

$$
\Big(\sum_{q\geq -1}2^{2q(s+1)}\|R^2_j\|^2_{L^1_T(L^2)}\Big)^{\frac12}\lesssim\int_0^t(1+\|a\|_{L^\infty})\big(\|\nabla a\|_{L^\infty}\|u\|_{H^{s+1}}+\|\nabla u\|_{L^\infty}\|a\|_{H^{s+1}}\big)\,dt'.
\tag{4.15}
$$

It remains to handle the pressure function Π in (4.1). Toward this, we get, by taking divergence to the momentum equation of (4.1), that

$$
\begin{aligned}
\mathrm{div}\{(1+a)\nabla\Pi\}=&-\mathrm{div}\{(u\cdot\nabla)u\}+\mathrm{div}\{a\,\mathrm{div}[(\widetilde\mu(a)-1)(2d)]\}\\
&+\mathrm{div}\,\mathrm{div}\{(\widetilde\mu(a)-1)(2d)\}+\mathrm{div}(a\Delta u),
\end{aligned}
\tag{4.16}
$$

applying Bony's decomposition (A.5) in the inhomogeneous contaxt to the right hand side of (4.16), we have

$$
\begin{aligned}
\operatorname{div}\{(1+a)\nabla\Pi\} = &-\operatorname{div}\{(u\cdot\nabla)u\} + \operatorname{div} T_a \operatorname{div}\big\{T_{\widetilde{\mu}(a)-1}(2d) + R(\widetilde{\mu}(a)-1, 2d)\big\}\\
&+ \operatorname{div} R(a, \operatorname{div}[(\widetilde{\mu}(a)-1)(2d)]) + \operatorname{div}\operatorname{div} T_{\widetilde{\mu}(a)-1}(2d)\\
&+ \operatorname{div}\operatorname{div} R(\widetilde{\mu}(a)-1, 2d) + T_{\nabla a}\Delta u + \operatorname{div} R(a, \Delta u),
\end{aligned}
$$

from which and the fact that $\operatorname{div} u = 0$, we infer

$$
\begin{aligned}
\operatorname{div}\{(1+a)\nabla\Pi\} = &-\operatorname{div}\{(u\cdot\nabla)u\} + T_{\nabla a}\operatorname{div}\big\{T_{\widetilde{\mu}(a)-\mu^1}(2d) + R(\widetilde{\mu}(a)-1, 2d)\big\}\\
&+ T_a \operatorname{div} T_{\nabla\widetilde{\mu}(a)}(2d) + T_a T_{\nabla\widetilde{\mu}(a)}\Delta u + T_a \operatorname{div}\operatorname{div} R(\widetilde{\mu}(a)-1, 2d)\\
&+ \operatorname{div} R(a, \operatorname{div}[(\widetilde{\mu}(a)-1)(2d)]) + \operatorname{div} T_{\nabla\widetilde{\mu}(a)}(2d) + T_{\nabla\widetilde{\mu}(a)}\Delta u\\
&+ \operatorname{div}\operatorname{div} R(\widetilde{\mu}(a)-1, 2d) + T_{\nabla a}\Delta u + \operatorname{div} R(a, \Delta u).
\end{aligned}
$$

taking L^2 inner product of the above equation with Π and using (4.4), we reach

$$
\begin{aligned}
\|\nabla\Pi\|_{L^2} \lesssim &\|\nabla u\|_{L^\infty}\|u\|_{L^2} + \|T_{\nabla a}\operatorname{div}\big\{T_{\widetilde{\mu}(a)-\mu^1}(2d) + R(\widetilde{\mu}(a)-\mu^1, 2d)\big\}\|_{\dot{H}^{-1}}\\
&+ \|T_a \operatorname{div} T_{\nabla\widetilde{\mu}(a)}(2d)\|_{\dot{H}^{-1}} + \|T_a T_{\nabla\widetilde{\mu}(a)}\Delta u\|_{\dot{H}^{-1}} + \|T_a \operatorname{div}\operatorname{div} R(\widetilde{\mu}(a)-1, 2d)\|_{\dot{H}^{-1}}\\
&+ \|R(a, \operatorname{div}[(\widetilde{\mu}(a)-1)(2d)])\|_{L^2} + \|T_{\nabla\widetilde{\mu}(a)}(2d)\|_{L^2} + \|T_{\nabla\widetilde{\mu}(a)}\Delta u\|_{\dot{H}^{-1}}\\
&+ \|R(\widetilde{\mu}(a)-1, 2d)\|_{\dot{H}^1} + \|T_{\nabla a}\Delta u\|_{\dot{H}^{-1}} + \|R(a, \Delta u)\|_{L^2},
\end{aligned}
$$

which together with standard para-product estimates ([6]) and (4.4) implies

$$
\|\nabla\Pi\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}(\|u\|_{L^2} + \|a\|_{H^1}) + \|\nabla a\|_{L^\infty}(1 + \|a\|_{H^1})\|\nabla u\|_{L^2} + \|a\|_{H^2}\|\nabla u\|_{L^2}. \tag{4.17}
$$

To deal with $\|\nabla\Pi\|_{H^{s-1}}$, we get by acting Δ_q to (4.16) and taking L^2 inner product of the resulting equation with $\Delta_q\Pi$ that

$$
\begin{aligned}
\|\nabla\Pi\|_{H^{s-1}} \lesssim &\|u\otimes u\|_{H^s} + \|(1+a)\operatorname{div}[(\widetilde{\mu}(a)-1)(2d)]\|_{H^{s-1}}\\
&+ \|(1+a)(2d)\|_{H^{s-1}} + \Big(\sum_{q\geq -1} 2^{2q(s-1)}\|[\Delta_q; a]\nabla\Pi\|_{L^2}^2\Big)^{\frac{1}{2}},
\end{aligned}
$$

from which, standard product laws in Sobolev space and Lemma A.2, we obtain

$$
\begin{aligned}
\|\nabla\Pi\|_{H^{s-1}} \lesssim &(1 + \|a\|_{L^\infty} + \|u\|_{L^\infty})\|u\|_{H^s} + \|a\|_{H^s}(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^\infty}) + \|a\|_{L^\infty}\|u\|_{H^{s+1}}\\
&+ \|a\|_{H^s}\big(\|a\|_{H^1}\|\nabla u\|_{L^\infty} + \|a\|_{L^\infty}\|u\|_{H^2}\big) + \|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{H^{s-2}} + \|\nabla\Pi\|_{L^2}\|a\|_{H^s}.
\end{aligned}
$$

If $s-2\leq 0$, then $\|\nabla\Pi\|_{H^{s-2}} \lesssim \|\nabla\Pi\|_{L^2}$ otherwise

$$
\|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{H^{s-2}} \leq \eta\|\nabla\Pi\|_{H^{s-1}} + C\|\nabla a\|_{L^\infty}^{s-1}\|\nabla\Pi\|_{L^2}.
$$

As a consequence, by taking η sufficiently small, we arrive at

$$
\begin{aligned}
\|\nabla\Pi\|_{H^{s-1}} \lesssim &(1 + \|u\|_{L^\infty})\|u\|_{H^s} + \|a\|_{H^s}\|\nabla u\|_{L^\infty} + \|a\|_{H^s}\big(\|a\|_{H^1}\|\nabla u\|_{L^\infty} + \|u\|_{H^2}\big)\\
&+ \|u\|_{H^{s+1}} + \big(\|\nabla a\|_{L^\infty} + \big\langle\big\langle\|\nabla a\|_{L^\infty}^{s-1}\big\rangle\big\rangle_{s>2} + \|a\|_{H^s}\big)\|\nabla\Pi\|_{L^2},
\end{aligned} \tag{4.18}
$$

where $\big\langle\big\langle\|\nabla a\|_{L^\infty}^{s-1}\big\rangle\big\rangle_{s>2} = \|\nabla a\|_{L^\infty}^{s-1}$ when $s>2$, and equal to 0 otherwise.

Therefore, substituting (4.11), (4.12), (4.14), (4.15), (4.17) and (4.18) into (4.10), we reach

$$
\begin{aligned}
\|u\|_{\widetilde{L}^\infty_T(H^s)}+\|u\|_{\widetilde{L}^1_T(H^{s+2})}\lesssim&\|u_0\|_{H^s}+\|u\|_{L^1_t(L^2)}+\int_0^t\|\nabla u\|_{L^\infty}\|u\|_{H^s}\,dt'\\
&+\int_0^t\|\nabla a\|_{L^\infty}\bigl\{(1+\|u\|_{L^\infty})\|u\|_{H^s}+\|a\|_{H^s}\bigl((1+\|a\|_{H^1})\|\nabla u\|_{L^\infty}+\|u\|_{H^2}\bigr)\\
&+\|u\|_{H^{s+1}}\bigr\}\,dt'+\int_0^t\bigl(\|\nabla a\|_{L^\infty}^2+\|\nabla a\|_{L^\infty}^s+\|a\|_{H^{s+1}}\bigr)\|\nabla\Pi\|_{L^2}\,dt'\\
&+\int_0^t\bigl\{\|\nabla a\|_{L^\infty}(\|\nabla u\|_{L^\infty}\|a\|_{H^s}+\|u\|_{H^{s+1}})+\|\nabla u\|_{L^\infty}\|a\|_{H^{s+1}}\bigr\}\,dt'.
\end{aligned}\tag{4.19}
$$

Thanks to (4.7), one has

$$
\begin{aligned}
\|a\|_{L^\infty_t(H^{s+1})}&\leq C(1+\|u\|_{L^1_t(H^{s+1})})\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr)\\
&\leq C\bigl(1+\|u\|_{L^1_t(H^1)}^{\frac{1}{s+1}}\|u\|_{\widetilde{L}^1_t(H^{s+2})}^{\frac{s}{s+1}}\bigr)\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr).
\end{aligned}
$$

It follows the same line that

$$
\|a\|_{L^\infty_t(H^s)}\leq C\bigl(1+\|u\|_{L^1_t(H^1)}^{\frac{2}{s+1}}\|u\|_{\widetilde{L}^1_t(H^{s+2})}^{\frac{s-1}{s+1}}\bigr)\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr),
$$

from which and (4.6), we deduce that

$$
\begin{aligned}
&\int_0^t\|\nabla a\|_{L^\infty}\|a\|_{H^s}\|u\|_{H^2}\,dt'\lesssim\bigl(1+t^{\frac{2}{s+2}}\|u\|_{\widetilde{L}^1_t(H^{s+2})}^{\frac{s}{s+2}}\bigr)\|u\|_{L^1_t(H^1)}^{\frac{s}{s+1}}\|u\|_{\widetilde{L}^1_t(H^{s+2})}^{\frac{1}{s+1}}\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr),\\
&\int_0^t\bigl(\|\nabla a\|_{L^\infty}^2+\|\nabla a\|_{L^\infty}^s\bigr)\|a\|_{H^2}\|\nabla u\|_{L^2}\,dt'\lesssim\sqrt{t}\bigl(1+\|u\|_{L^1_t(H^1)}^{\frac{s}{s+1}}\|u\|_{\widetilde{L}^1_t(H^{s+2})}^{\frac{1}{s+1}}\bigr)\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr),\\
&\int_0^t\|a\|_{H^{s+1}}\|\nabla u\|_{L^\infty}\,dt'\lesssim\bigl(1+\|u\|_{L^1_\tau(H^1)}^{\frac{1}{s+1}}\|u\|_{\widetilde{L}^1_\tau(H^{s+2})}^{\frac{s}{s+1}}\bigr)\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t\|\nabla u\|_{L^2}\|a\|_{H^{s+1}}\|a\|_{H^2}\,dt'\\
&\quad\lesssim\int_0^t\|\nabla u\|_{L^2}(1+\|u\|_{L^1_{t'}(H^{s+1})})(1+\|u\|_{L^1_t(H^2)})\exp\bigl(C\|\nabla u\|_{L^1_{t'}(L^\infty)}\bigr)\,dt'\\
&\quad\lesssim\int_0^t\|\nabla u\|_{L^2}\bigl(1+\|u\|_{L^1_\tau(H^1)}^{\frac{1}{s+1}}\|u\|_{\widetilde{L}^1_\tau(H^{s+2})}^{\frac{s}{s+1}}+\|u\|_{L^1_\tau(H^1)}^{\frac{s}{s+1}}\|u\|_{\widetilde{L}^1_\tau(H^{s+2})}^{\frac{1}{s+1}}\\
&\qquad+\|u\|_{\widetilde{L}^1_\tau(H^{s+2})}\bigr)\exp\bigl(C\|\nabla u\|_{L^1_{t'}(L^\infty)}\bigr)\,dt'.
\end{aligned}
$$

Substituting the above inequalities into (4.19) and using Young inequality, we obtain

$$
\begin{aligned}
\|u\|_{\widetilde{L}^\infty_t(H^s)}+\|u\|_{\widetilde{L}^1_t(H^{s+2})}\lesssim&1+t+\int_0^t\|\nabla u\|_{L^2}\exp\bigl(C\|\nabla u\|_{L^1_{t'}(L^\infty)}\bigr)\|u\|_{\widetilde{L}^1_\tau(H^{s+2})}\,dt'\\
&+\int_0^t\bigl(\|\nabla u\|_{L^\infty}+(1+\sqrt{t'})\exp\bigl(C\|\nabla u\|_{L^1_{t'}(L^\infty)}\bigr)\bigr)\|u\|_{H^s}\,dt'\\
&+f(t,s,a_0,u_0)\exp\bigl(C\|\nabla u\|_{L^1_t(L^\infty)}\bigr),
\end{aligned}\tag{4.20}
$$

for $f(t,s,a_0,u_0)$ given by

$$
f(t,s,u_0,a_0)\stackrel{\mathrm{def}}{=}1+t+\|u_0\|_{H^s}+\bigl(1+t^{\frac{s+1}{2}}+t^{\frac{s+1}{2s}}\bigr)\|u\|_{L^1_t(H^1)}+\bigl(t^{\frac{2s+2}{s}}+t^{\frac{6+s}{2s}}\bigr)\|u\|_{L^1_t(H^1)}^{s+2}.
$$

Applying Gronwall's inequality to (4.20) and using (4.5), we arrive at

$$\|u\|_{\widetilde{L}^\infty_T(H^s)}+\|u\|_{\widetilde{L}^1_T(H^{s+2})}\leq C\,f(t,s,u_0,a_0)\exp\Big\{C(1+\sqrt{t}+t^{\frac32})\exp\big\{C\|\nabla u\|_{L^1_t(L^\infty)}\big\}\Big\}.$$

This together with (4.7) completes the proof of Theorem 4.1. □

5. The proof of Theorem 1.2

Notice from [15] that with the additional regularity assumptions that $\nabla\rho_0\in L^r(\mathbb{T}^2)$ for some $r>2$, Desjardins proved that: there exists some positive time τ so that Lions weak solution (ρ,u) satisfies $u\in L^2((0,\tau);H^2(\mathbb{R}^2))$ and $\nabla\rho\in L^\infty((0,\tau);L^{\bar r}(\mathbb{T}^2))$ for any $\bar r<r$. Here with Proposition 3.1, we shall prove that $\tau=\infty$ and $\bar r=r$.

Proposition 5.1. *Let $(\rho,u,\nabla\Pi)$ be a smooth enough solution of (1.1) on $[0,T^*)$. Then under the assumptions of Theorem 1.2, we have*

$$\begin{aligned}&\|\nabla u\|_{L^1_t(L^\infty)}\leq 2CC_1^2\big(1+\|\rho_0\|_{B^{(2/q)+}_{\infty,\infty}}\big)\overset{\text{def}}{=}\mathfrak{C},\\ &\|\nabla\rho\|_{L^\infty_t(L^r)}+\|\Delta u\|_{L^1_t(L^2)}+\|\nabla\Pi\|_{L^1_t(L^2)}\leq C\big(1+\|\nabla\rho_0\|_{L^r}\big)\exp\big(C\mathfrak{C}\big)\end{aligned}\tag{5.1}$$

for $t\leq T^*$ and C_1 given by (2.20). Here and in that follows, the uniform constant C may depends on m, M and $\|\mu'\|_{L^\infty}$.

Proof. Under the smallness assumption (1.8), we find that (3.9) is satisfied. Hence, we get, by applying Proposition 3.1, that

$$\|\nabla u\|_{L^\infty_t(L^\infty)}\leq\|u\|_{L^1_t(\dot B^1_{\infty,1})}\leq 2CC_1^2\big(1+\|\rho_0\|_{B^{\frac{2}{q}+\varepsilon}_{\infty,\infty}}\big)\quad\text{for}\quad t<T^*\text{ and any }\ \varepsilon>0.\tag{5.2}$$

On the other hand, by taking L^2 inner product of Δu with the momentum equation of (1.1), we obtain

$$\|\sqrt{\mu(\rho)}\Delta u\|_{L^2}^2=\int_{\mathbb{R}^2}\big(\rho\partial_t u+\rho(u\cdot\nabla)u-2\mu'(\rho)\nabla\rho\cdot d\big)\ |\ \Delta u\,dx,$$

from which and (2.2), we infer

$$\|\Delta u\|_{L^1_t(L^2)}\leq C\Big(\|\partial_t u\|_{L^1_t(L^2)}+\|\nabla\rho\|_{L^\infty_t(L^r)}\|\nabla u\|_{L^1_t(L^n)}+\|u\cdot\nabla u\|_{L^1_t(L^2)}\Big),\tag{5.3}$$

where n is determined by $\frac1r+\frac1n=\frac12$. It is easy to check, from the transport equation of (1.1) and (5.2), that

$$\|\nabla\rho\|_{L^\infty_t(L^r)}\leq\|\nabla\rho_0\|_{L^r}\exp\big(C\|\nabla u\|_{L^1_t(L^\infty)}\big)\leq\|\nabla\rho_0\|_{L^r}\exp(C\mathfrak{C})\quad\text{for}\quad t\leq T^*,$$

moreover, as $r<\frac{2}{1-2\delta}$, (2.21) ensures that

$$\|\nabla u\|_{L^1_t(L^n)}\leq C\|\langle t'\rangle^{(\frac12+\delta-\frac1n)-}\nabla u\|_{L^2_t(L^2)}\|\langle t'\rangle^{-(\delta+\frac1r)-}\|_{L^2_t}\leq CC_1^2,$$

so that we deduce from (5.3) that

$$\|\Delta u\|_{L^2_t(L^2)}\leq C\big(1+\|\nabla\rho_0\|_{L^r}\big)\exp\big(C\mathfrak{C}\big)\quad\text{for}\quad t<T^*.\tag{5.4}$$

Finally thanks to the momentum equation of (1.1), one has

$$\|\nabla\Pi\|_{L^1_t(L^2)}\leq C\Big(\|\partial_t u\|_{L^1_t(L^2)}+\|\nabla\rho\|_{L^\infty_t(L^r)}\|\nabla u\|_{L^1_t(L^n)}+\|u\cdot\nabla u\|_{L^1_t(L^2)}+\|\Delta u\|_{L^1_t(L^2)}\Big),$$

which along with the proof of (5.4) leads to the estimate of $\|\nabla\Pi\|_{L^1_t(L^2)}$. This completes the proof of Proposition 5.1. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Firstly under the assumption of (1.6) and (1.8), Theorem 4.1 together with Proposition 5.1 ensures (1.1) has a global solution (ρ, u) with $\rho - 1 \in \mathcal{C}([0,\infty); H^{1+s}(\mathbb{R}^2))$ and $u \in \mathcal{C}([0,\infty); H^s(\mathbb{R}^2)) \cap \widetilde{L}^1_{\rm loc}(\mathbb{R}^+; H^{2+s}(\mathbb{R}^2))$ provided that $\rho_0 - 1 \in H^{1+s}(\mathbb{R}^2)$, $u_0 \in H^s(\mathbb{R}^2)$ and $\mu(\cdot) \in W^{2+[s],\infty}(\mathbb{R}^+)$ for some $s > 1$.

To prove the global existence of strong solutions of (1.1) without the additional regularity assumption that $\rho_0 - 1 \in H^{1+s}(\mathbb{R}^2)$ and $u_0 \in H^s(\mathbb{R}^2)$ for $s > 1$, we denote $\rho_{0,\eta} \stackrel{\text{def}}{=} \rho_0 * j_\eta$, $u_{0,\eta} \stackrel{\text{def}}{=} u_0 * j_\eta$, and $\mu_\eta = \mu * j_\eta$, where $j_\eta(|x|) = \eta^{-2} j(|x|/\eta)$ is the standard Friedrich's mollifier. Then (1.1) with viscous coefficient μ_η and with initial data $(\rho_{0,\eta}, u_{0,\eta})$ has a global solution $(\rho_\eta, u_\eta, \nabla\Pi_\eta)$. Moreover, Proposition 5.1 ensures that $(\rho_\eta, u_\eta, \nabla\Pi_\eta)$ satisfy the uniform estimates (2.10) and (5.1). This together with a standard compactness argument yields the existence part of Theorem 1.2. For simplicity, we skip the details here.

It remains to prove the uniqueness part of Theorem 1.2. Indeed let $(\rho_i, u_i, \nabla\Pi_i)$, for $i = 1, 2$, be two solutions of (1.1) so that $\rho_i \in \mathcal{C}_b([0,T]; L^\infty \cap \dot{W}^{1,r}(\mathbb{R}^2))$, $u_i \in \mathcal{C}_b([0,T]; L^2(\mathbb{R}^2)) \cap L^2((0,T); \dot{H}^1(\mathbb{R}^2)) \cap L^1((0,T); \mathrm{Lip}(\mathbb{R}^2))$, and $\partial_t u \in L^2((0,T); L^2(\mathbb{R}^2))$, we denote by

$$(\delta\rho, \delta u, \nabla\delta\Pi) \stackrel{\text{def}}{=} (\rho_2 - \rho_1, u_2 - u_1, \nabla\Pi_2 - \nabla\Pi_1). \tag{5.5}$$

Then the system for $(\delta\rho, \delta u, \delta\nabla\Pi)$ reads

$$\begin{cases} \partial_t \delta\rho + u_2 \cdot \nabla \delta\rho = -\delta u \cdot \nabla\rho_1 \\ \rho_2 \partial_t \delta u + \rho_2 (u_2 \cdot \nabla)\delta u - 2\mathrm{div}\{\mu(\rho_2) d(\delta u)\} + \nabla\delta\Pi = \delta F, \\ \mathrm{div}\, \delta u = 0, \\ (\delta\rho, \delta u)_{|t=0} = (0,0). \end{cases} \tag{5.6}$$

where δF is determined by

$$\delta F = -\delta\rho\partial_t u_1 - \delta\rho(u_2 \cdot \nabla)u_1 - \rho_1(\delta u \cdot \nabla)u_1 + 2\mathrm{div}\{(\mu(\rho_2) - \mu(\rho_1)) d(u_1)\}.$$

Let $2 < m < r$, and $p \stackrel{\text{def}}{=} \frac{mr}{r-m}$, we deduce from the transport equation of (5.6) that

$$\begin{aligned} \|\delta\rho(t)\|_{L^m} \leq & \int_0^t \|\delta u\|_{L^p} \|\nabla\rho_1\|_{L^r}\, dt' \leq \|\nabla\rho_1\|_{L^\infty_t(L^r)} \|\delta u\|_{L^1_t(L^p)} \\ \leq & C t^{\frac12 + \frac1p} \|\nabla\rho_1\|_{L^\infty_t(L^r)} \|\delta u\|^{\frac2p}_{L^\infty_t(L^2)} \|\nabla\delta u\|^{1-\frac2p}_{L^2_t(L^2)}. \end{aligned} \tag{5.7}$$

Whereas taking L^2 inner product δu with the momentum equation of (5.6), we get

$$\frac12 \frac{d}{dt} \int_{\mathbb{R}^2} \rho_2 |\delta u|^2\, dx + 2\int_{\mathbb{R}^2} \mu(\rho_2) d(\delta u) : d(\delta u)\, dx = \int_{\mathbb{R}^2} \delta F \mid \delta u\, dx,$$

which leads to

$$\begin{aligned} \|\delta u\|^2_{L^\infty_t(L^2)} + \|\nabla\delta u\|^2_{L^2_t(L^2)} \leq & \|\delta\rho\|_{L^\infty_t(L^m)} \int_0^t \big(\|\partial_t u_1\|_{L^2} + \|u_2\|_{L^4} \|\nabla u_1\|_{L^4}\big) \|\delta u\|_{L^{\bar m}}\, dt' \\ & + \int_0^t \|\rho_1\|_{L^\infty} \|\nabla u_1\|_{L^\infty} \|\delta u\|^2_{L^2}\, dt' \\ & + \|\delta\rho\|_{L^\infty_t(L^m)} \int_0^t \|\nabla u_1\|_{L^{\bar m}} \|\nabla\delta u\|_{L^2}\, dt', \end{aligned} \tag{5.8}$$

where $\frac1m + \frac1{\bar m} = \frac12$. It follows from (5.7) that

$$\begin{aligned} \|\delta\rho\|_{L^\infty_t(L^m)} \int_0^t \|\partial_t u_1\|_{L^2} \|\delta u\|_{L^{\bar m}}\, dt' \leq & C t^{1-\frac1r} \|\partial_t u_1\|_{L^2_t(L^2)} \|\delta u\|^{2(\frac12 - \frac1r)}_{L^\infty_t(L^2)} \|\nabla\delta u\|^{2(\frac12+\frac1r)}_{L^2_t(L^2)} \\ \leq & \eta_1(t) \big(\|\delta u\|^2_{L^\infty_t(L^2)} + \|\nabla\delta u\|^2_{L^2_t(L^2)}\big), \end{aligned}$$

where $\lim_{t\to 0} \eta_1(t) = 0$. The same estimate holds for $\|\delta\rho\|_{L^\infty_t(L^m)} \int_0^t \|u_2\|_{L^4} \|\nabla u_1\|_{L^4} \|\delta u\|_{L^{\bar m}}\, dt'$.

While again notice from (5.7) that

$$\begin{aligned}\|\delta\rho\|_{L^\infty_t(L^m)}&\int_0^t\|\nabla u_1\|_{L^{\bar m}}\|\nabla\delta u\|_{L^2}\,dt'\\&\leq Ct^{1+\frac{2}{q}}\|\nabla u_1\|^2_{L^2_t(L^{\bar m})}\|\delta u\|^{\frac{4}{p}}_{L^\infty_t(L^2)}\|\nabla\delta u\|^{2(1-\frac{2}{p})}_{L^2_t(L^2)}+\frac{1}{8}\|\nabla\delta u\|^2_{L^2_t(L^2)}\\&\leq\eta_2(t)\|\delta u\|^2_{L^\infty_t(L^2)}+\frac{1}{4}\|\nabla\delta u\|^2_{L^2_t(L^2)},\end{aligned}$$

where $\eta_2(t)$ satisfies $\lim_{t\to0}\eta_2(t)=0$.

Hence taking t_1 small enough, we infer from (5.8) that

$$\|\delta u\|^2_{L^\infty_t(L^2)}\leq C\int_0^t\|\rho_1\|_{L^\infty}\|\nabla u_1\|_{L^\infty}\|\delta u\|^2_{L^2}\,dt'\quad\text{for}\quad t\leq t_1.$$

Applying Gronwall's inequality yields

$$\delta u=0\quad\text{for}\quad t\leq t_1,$$

from which and (5.7), we obtain $\delta\rho=0$ for $t\leq t_1$. Finally thanks to the momentum equation of (5.6), we get that $\nabla\delta\Pi=0$ for $t\leq t_1$. The uniqueness on the whole time interval $[0,T]$ then follows by a bootstrap argument. This completes the proof of Theorem 1.2. $\square$

6. Proof of Corollary 1.1

In this section, we shall repeat the arguments from Section 2, Section 3 and Section 5 to prove the global well-posedness of (1.1) in the case of constant viscosity.

Proof of Corollary 1.1. We first deduce, by a similar proof of Theorem 1 of [4], that the system (1.1) with $\mu(\rho)=1$ has a unique local solution on $[0,T^*)$ so that

$$\begin{aligned}&\rho-1\in\mathcal{C}_b([0,t];\dot B^1_{2,1}\cap\dot B^\alpha_{\infty,\infty}(\mathbb{R}^2))\quad\text{and}\quad u\in\mathcal{C}_b([0,t];\dot B^0_{2,1}(\mathbb{R}^2))\cap L^1([0,t];\,\dot B^2_{2,1}(\mathbb{R}^2)),\\&\frac12\|\sqrt{\rho}u(t)\|^2_{L^2}+\|\nabla u\|^2_{L^2_t(L^2)}=\frac12\|\sqrt{\rho_0}u_0\|^2_{L^2}\quad\text{and}\quad m\leq\rho(t,x)\leq M,\end{aligned}\tag{6.1}$$

for any $t<T^*$. Then to complete the proof of Corollary 1.1, we only need to show that $T^*=\infty$.

In fact, thanks to (6.1), we can find some $t_0\in(0,T^*)$ such that $u(t_0)\in H^1(\mathbb{R}^2)$. Then for $t_0\leq t<T^*$, we get, by taking the L^2 inner product of the momentum equation of (1.1) with $\partial_t u$, that

$$\begin{aligned}\frac12\|\nabla u(t)\|^2_{L^2}+&\|\sqrt{\rho}\partial_t u\|^2_{L^2((t_0,t);L^2)}\\&=\frac12\|\nabla u(t_0)\|^2_{L^2}-\int_{t_0}^t\int_{\mathbb{R}^2}\rho u\cdot\nabla u\mid\partial_t u\,dx\,dt'\\&\leq\frac12\|\nabla u(t_0)\|^2_{L^2}+\frac{M^2}{2m}\int_{t_0}^t\|u\|^2_{L^4}\|\nabla u\|^2_{L^4}\,dt'+\frac{m}{2}\|\partial_t u\|^2_{L^2((t_0,t);L^2)},\end{aligned}$$

which along with (6.1) and $\|a\|^2_{L^4}\leq C\|u\|_{L^2}\|\nabla u\|_{L^2}$ implies that for any $\varepsilon>0$ and $t<T^*$,

$$\|\nabla u(t)\|^2_{L^2}+m\|\partial_t u\|^2_{L^2((t_0,t);L^2)}\leq\|\nabla u(t_0)\|^2_{L^2}+\frac{M^4}{4m^2\varepsilon}\int_{t_0}^t\|u\|^2_{L^2}\|\nabla u\|^4_{L^2}\,dt'+\varepsilon\|\Delta u\|^2_{L^2((t_0,t);L^2)}.\tag{6.2}$$

However, when $\mu(\rho)=1$, we deduce from the property of linear Stokes system that

$$\begin{aligned}\|\Delta u\|^2_{L^2((t_0,t);L^2)}+\|\nabla\Pi\|^2_{L^2((t_0,t);L^2)}&\leq\|\rho\partial_t u\|^2_{L^2((t_0,t);L^2)}+\|\rho u\cdot\nabla u\|^2_{L^2((t_0,t);L^2)}\\&\leq M^2\|\partial_t u\|^2_{L^2((t_0,t);L^2)}+\frac{M^4}{2}\int_{t_0}^t\|u\|^2_{L^2}\|\nabla u\|^4_{L^2}\,dt'+\frac12\|\Delta u\|^2_{L^2((t_0,t);L^2)},\end{aligned}$$

which gives rise to

$$\|\Delta u\|_{L^2((t_0,t);L^2)}^2+\|\nabla\Pi\|_{L^2((t_0,t);L^2)}^2\leq 2M^2\|\partial_t u\|_{L^2((t_0,t);L^2)}^2+M^4\int_{t_0}^t\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4\,dt'. \tag{6.3}$$

Summing up (6.2) with $2\varepsilon\times$(6.3), we obtain

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2+&m\|\partial_t u\|_{L^2((t_0,t);L^2)}^2+\varepsilon\big(\|\Delta u\|_{L^2((t_0,t);L^2)}^2+\|\nabla\Pi\|_{L^2((t_0,t);L^2)}^2\big)\\&\leq\|\nabla u(t_0)\|_{L^2}^2+4M^2\varepsilon\|\partial_t u\|_{L^2((t_0,t);L^2)}^2+C_{m,M}\|u_0\|_{L^2}^2\int_{t_0}^t\|\nabla u\|_{L^2}^4\,dt'.\end{aligned}$$

Taking $\varepsilon=\frac{m}{8M^2}$ in the above inequality and then applying Gronwall's inequality leads to

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\frac{m}{2}\|\partial_t u\|_{L^2((t_0,t);L^2)}^2+\frac{m}{8M^2}\big(\|\Delta u\|_{L^2((t_0,t);L^2)}^2+\|\nabla\Pi\|_{L^2((t_0,t);L^2)}^2\big)\\&\leq\|\nabla u(t_0)\|_{L^2}^2\exp\big\{C_{m,M}\|u_0\|_{L^2}^2\|\nabla u\|_{L^2((t_0,t);L^2)}^2\big\}\\&\leq\|\nabla u(t_0)\|_{L^2}^2\exp\big\{C_{m,M}\|u_0\|_{L^2}^4\big\}.\end{aligned}\tag{6.4}$$

On the other hand, we get, by a similar derivation of (3.4), that

$$\begin{aligned}\|u\|_{\widetilde{L}^1((t_0,t);\dot B^{2+\alpha}_{2,\infty})}\lesssim&\|u(t_0)\|_{H^1}+\sup_j 2^{j\alpha}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1((t_0,t);L^2)}\\&+\sup_j 2^{j\alpha}\|[\dot\Delta_j\mathbb{P};\frac{1}{\rho}]\big(\Delta u-\nabla\Pi\big)\|_{L^1((t_0,t);L^2)}.\end{aligned}\tag{6.5}$$

The proof of (3.5) yields

$$\sup_j 2^{j\alpha}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1((t_0,t);L^2)}\lesssim\|\nabla u\|_{L^2((t_0,t);L^2)}^{2-\alpha}\|\Delta u\|_{L^2((t_0,t);L^2)}^{\alpha}.$$

And it follows form the proof of (3.7) that

$$\sup_j 2^{j\alpha}\|[\dot\Delta_j\mathbb{P};\frac{1}{\rho}]\big(\Delta u-\nabla\Pi\big)\|_{L^1((t_0,t);L^2)}\lesssim\sqrt{t-t_0}\;\|\rho\|_{L^\infty((t_0,t);\dot B^\alpha_{\infty,\infty})}\|\Delta u-\nabla\Pi\|_{L^2((t_0,t);L^2)}.$$

Therefore thanks to (6.4) and (6.5), we conclude that

$$\begin{aligned}\|u\|_{\widetilde{L}^1((t_0,t);\dot B^{2+\alpha}_{2,\infty})}\leq&C(m,M,\|u(t_0)\|_{H^1})\big(1+\sqrt{t-t_0}\;\|\rho\|_{L^\infty((t_0,t);\dot B^\alpha_{\infty,\infty})}\big)\\\leq&C(m,M,\|u(t_0)\|_{H^1})\Big\{1+\sqrt{t-t_0}\;\|\rho(t_0)\|_{\dot B^\alpha_{\infty,\infty}}\exp\big(C\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\big)\Big\}.\end{aligned}\tag{6.6}$$

Now for any positive integer N, we get, by applying (A.2), that

$$\begin{aligned}\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\leq&\sum_{\ell\leq0}2^\ell\|\dot\Delta_\ell\nabla u\|_{L^1((t_0,t);L^2)}+\sum_{0<\ell\leq N}\|\dot\Delta_\ell\Delta u\|_{L^1((t_0,t);L^2)}\\&+\sum_{\ell\geq N}2^{2\ell}\|\dot\Delta_\ell u\|_{L^1((t_0,t);L^2)}\\\lesssim&\sqrt{t-t_0}\;\|\nabla u\|_{L^2((t_0,t);L^2)}+\sqrt{N(t-t_0)}\;\|\Delta u\|_{L^2((t_0,t);L^2)}\\&+2^{-N\alpha}\|u\|_{\widetilde{L}^1((t_0,t);\dot B^{2+\alpha}_{2,\infty})},\end{aligned}$$

which together with (6.1),(6.4) and (6.6) implies

$$\begin{aligned}\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\leq&C(m,M,\|\rho(t_0)\|_{\dot B^\alpha_{\infty,\infty}},\|u(t_0)\|_{H^1})\Big\{1\\&+\sqrt{t-t_0}\big(1+\sqrt{N}+2^{-N\alpha}\exp\big(C\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\big)\big)\Big\}.\end{aligned}$$

Taking $N=\left[\frac{C}{\alpha}\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\right]$ in the above inequality results in

$$\begin{aligned}\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\leq&C(m,M,\|\rho(t_0)\|_{\dot B^\alpha_{\infty,\infty}},\|u(t_0)\|_{H^1})\Big\{1+\sqrt{t-t_0}\big(1+\sqrt{\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}}\big)\Big\}\\ \leq&C(m,M,\|\rho(t_0)\|_{\dot B^\alpha_{\infty,\infty}},\|u(t_0)\|_{H^1})(1+t-t_0)+\frac12\|u\|_{L^1((t_0,t);\dot B^2_{2,1})},\end{aligned}$$

from which, we infer

$$\|u\|_{L^1((t_0,t);\dot B^2_{2,1})}\leq C(m,M,\|\rho(t_0)\|_{\dot B^\alpha_{\infty,\infty}},\|u(t_0)\|_{H^1})(1+t-t_0). \tag{6.7}$$

With (6.7), it is standard to prove that T^* given at the beginning of the proof equals ∞. This completes the proof of the corollary. □

7. Proof of Theorem 1.3

The goal of this section is to present the proof of Theorem 1.3. To prove the existence part of Theorem 1.3, we need the following two technical lemmas:

Lemma 7.1. *Let $p,q\geq 1$ and $s\in\mathbb{R}$ satisfying $\frac1p\leq\frac12+\frac1q$ and $\max\big(-1,2(\frac1q-1)\big)<s<1+\frac2q$. Let $v\in\dot B^s_{q,2}\cap\dot H^1(\mathbb{R}^2)$ be a solenoidal vector filed. Then one has*

$$\|[\dot\Delta_j\mathbb{P};v\cdot\nabla]v\|_{L^p}\lesssim d_j2^{j(1+\frac2q-\frac2p-s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}}. \tag{7.1}$$

Proof. We get, by using Bony's decomposition (A.5), that

$$[\dot\Delta_j\mathbb{P};v\cdot\nabla]v=[\dot\Delta_j\mathbb{P};T_v\cdot\nabla]v+\dot\Delta_j\mathbb{P}\big(T_{\nabla v}v+\mathcal{R}(v,\nabla v)\big)-T_{\nabla\dot\Delta_jv}v-\mathcal{R}(v,\nabla\dot\Delta_jv). \tag{7.2}$$

It is easy to check that

$$[\dot\Delta_j\mathbb{P};T_v\cdot\nabla]v(x)=2^{2j}\sum_{|\ell-j|\leq4}\int_0^1\int_{\mathbb{R}^2}h(2^jz)z\cdot\dot S_{\ell-1}\nabla v(x+(\theta-1)z)\dot\Delta_\ell\nabla v(x-z)\,dz\,d\theta.$$

We first deal with the case when $1<p\leq2$ in (7.1). In this case, applying Hölder's inequality and the property of the translation invariance of the Lebesgue measure, we obtain

$$\begin{aligned}\|[\dot\Delta_j\mathbb{P};T_v\cdot\nabla]v\|_{L^p}\leq&2^j\sum_{|\ell-j|\leq4}\int_0^1\int_{\mathbb{R}^2}|h_1(2^jz)|\|\dot S_{\ell-1}\nabla v(\cdot+\theta z)\|_{L^{\bar p}}\|\dot\Delta_\ell\nabla v(\cdot-z)\|_{L^2}\,dzd\theta\\ \lesssim&2^{-j}\sum_{|\ell-j|\leq4}\|\dot S_{\ell-1}\nabla v\|_{L^{\bar p}}\|\dot\Delta_\ell\nabla v\|_{L^2},\end{aligned}$$

where $h_1(z)=zh(z)$ and $\bar p$ satisfies $\frac1p=\frac12+\frac1{\bar p}$. As $\frac1p\leq\frac12+\frac1q$ and $s<1+\frac2q$, we deduce, from Lemma A.1, that

$$\|\dot S_{\ell-1}\nabla v\|_{L^{\bar p}}\lesssim\sum_{k\leq\ell-1}2^{2k(\frac1q-\frac1{\bar p})}\|\dot\Delta_k\nabla v\|_{L^q}\lesssim c_\ell2^{\ell[2(1+\frac1q-\frac1p)-s]}\|v\|_{\dot B^s_{q,2}}$$

so that one has

$$\|[\dot\Delta_j\mathbb{P};T_v\cdot\nabla]v\|_{L^p}\lesssim d_j2^{j(1+\frac2q-\frac2p-s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}}. \tag{7.3}$$

Along the same line, we have

$$\|\dot\Delta_j\mathbb{P}\big(T_{\nabla v}v\big)\|_{L^p}\leq\sum_{|\ell-j|\leq4}\|\dot S_{\ell-1}\nabla v\|_{L^{\bar p}}\|\dot\Delta_\ell v\|_{L^2}\lesssim d_j2^{j(1+\frac2q-\frac2p-s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}}. \tag{7.4}$$

The same estimate holds for $T_{\nabla\dot\Delta_jv}v$.

Whereas for $1<q\leq 2$, we get, by applying $\operatorname{div} v=0$ and Lemma A.1, that

$$\begin{aligned}\|\dot\Delta_j\mathbb{P}(\mathcal{R}(v,\nabla v))\|_{L^p}&\lesssim 2^{j(3-\frac{2}{p})}\sum_{\ell\geq j-3}\|\dot\Delta_\ell v\|_{L^2}\|\widetilde{\dot\Delta}_\ell v\|_{L^2}\\&\lesssim 2^{j(3-\frac{2}{p})}\sum_{\ell\geq j-3}d_\ell 2^{\ell(\frac{2}{q}-s-2)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}}\lesssim d_j 2^{j(1+\frac{2}{q}-\frac{2}{p}-s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}},\end{aligned}\tag{7.5}$$

and for $q\geq 2$, we have

$$\begin{aligned}\|\dot\Delta_j\mathbb{P}(\mathcal{R}(v,\nabla v))\|_{L^p}&\lesssim 2^{j[2(1+\frac{1}{q}-\frac{1}{p})]}\sum_{\ell\geq j-3}\|\dot\Delta_\ell u\|_{L^q}\|\widetilde{\dot\Delta}_\ell v\|_{L^2}\\&\lesssim 2^{j[2(1+\frac{1}{q}-\frac{1}{p})]}\sum_{\ell\geq j-3}d_\ell 2^{-\ell(1+s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}}\lesssim d_j 2^{j(1+\frac{2}{q}-\frac{2}{p}-s)}\|\nabla v\|_{L^2}\|v\|_{\dot B^s_{q,2}},\end{aligned}\tag{7.6}$$

where we used the fact that $s>\max\big(-1,2(\frac{1}{q}-1)\big)$. This together with (7.3) and (7.4) proves (7.1) for $1<p\leq 2$.

The case when $2<p$ is much easier. Notice that

$$\|[\dot\Delta_j\mathbb{P};T_v\cdot\nabla]v\|_{L^p}\lesssim 2^{-j}\sum_{|\ell-j|\leq 4}\|\dot S_{\ell-1}\nabla v\|_{L^\infty}\|\dot\Delta\nabla v\|_{L^p},$$

and as $s<1+\frac{2}{q}$, one has

$$\|\dot S_{\ell-1}\nabla v\|_{L^\infty}\lesssim c_\ell 2^{\ell(1+\frac{2}{q}-s)}\|v\|_{\dot B^s_{q,2}},$$

so that (7.3) holds for $p>2$. The same estimate holds for $\dot\Delta_j\mathbb{P}\big(T_{\nabla v}v\big)$ and $T_{\nabla\dot\Delta_j v}v$. This together with (7.5) and (7.6) completes the proof of (7.1) for $2<p$. □

Lemma 7.2. *Let $p\geq 1$ and $s>-1$. Let $v\in\dot B^{1+s}_{p,1}\cap\dot B^{2+s}_{p,1}\cap H^1(\mathbb{R}^2)$ be a solenoidal vector filed. Then one has*

$$\|v\cdot\nabla v\|_{\dot B^s_{p,1}}\lesssim\|v\|_{L^2}\|v\|_{\dot B^{2+s}_{p,1}}+\|\nabla v\|_{L^2}\|v\|_{\dot B^{1+s}_{p,1}}.\tag{7.7}$$

Proof. Bony's decomposition (A.5) for $v\cdot\nabla v$ reads

$$v\cdot\nabla v=T_v\cdot\nabla v+T_{\nabla v}\cdot v+\mathcal{R}(v,\nabla v).$$

Applying Lemma A.1 yields

$$\|\dot\Delta_j(T_v\cdot\nabla v)\|_{L^p}\lesssim\sum_{|\ell-j|\leq 4}\|\dot S_{\ell-1}v\|_{L^2}\|\dot\Delta_\ell\nabla v\|_{L^{\bar p}}\lesssim d_j 2^{-js}\|v\|_{L^2}\|v\|_{\dot B^{2+s}_{p,1}},$$

where $\bar p$ satisfies $\frac{1}{p}=\frac{1}{2}+\frac{1}{\bar p}$.

A similar procedure gives rise to

$$\|\dot\Delta_j(T_{\nabla v}\cdot v)\|_{L^p}\lesssim\sum_{|\ell-j|\leq 4}\|\dot S_{\ell-1}\nabla v\|_{L^2}\|\dot\Delta_\ell v\|_{L^{\bar p}}\lesssim d_j 2^{-js}\|\nabla v\|_{L^2}\|v\|_{\dot B^{1+s}_{p,1}}.$$

Finally as $s>-1$, by applying $\operatorname{div} v=0$ and Lemma A.1, we get

$$\|\dot\Delta_j(\mathcal{R}(v,\nabla v)\|_{L^p}\lesssim 2^j\sum_{\ell\geq j-3}\|\dot\Delta_\ell v\|_{L^2}\|\widetilde{\dot\Delta}_\ell v\|_{L^{\bar p}}\lesssim d_j 2^{-js}\|v\|_{L^2}\|v\|_{\dot B^{2+s}_{p,1}}.$$

This completes the proof of (7.7). □

Proof to the existence part of Theorem 1.3. Given initial data (ρ_0, u_0) satisfying the assumptions of Theorem 1.3, we deduce from [2] that (1.1) has a local solution (ρ, u) on $[0, T^*)$ so that

$$\begin{aligned}&\rho-1\in\mathcal{C}_b([0,T];\dot B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)),\quad u\in\mathcal{C}_b([0,T];\dot B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap L^1([0,T];\,\dot B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\quad\text{and}\\&\frac12\|\sqrt{\rho}u(T)\|_{L^2}^2+\int_0^T\int_{\mathbb{R}^2}\mu(\rho)d(u):d(u)\,dx\,dt'=\frac12\|\sqrt{\rho_0}u_0\|_{L^2}^2\end{aligned}\tag{7.8}$$

for any $T<T^*$.

In order to prove that $T^*=\infty$ under the nonlinear smallness condition (1.11), we write

$$\partial_t u+(u\cdot\nabla)u-\Delta u+\nabla\Pi=(1-\rho)\partial_t u+(1-\rho)(u\cdot\nabla)u+\operatorname{div}\big[2(\mu(\rho)-1)d\big],\tag{7.9}$$

from which and similar derivation of (3.4), we deduce for $p\in(1,4)$ and $t\in(0,T^*)$ that

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+c\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\leq\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+\sum_{j\in\mathbb{Z}}2^{j(-1+\frac{2}{p})}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1_t(L^p)}&\\+\big\|\big\{(1-\rho)\partial_t u+(1-\rho)(u\cdot\nabla)u+\operatorname{div}\big[2(\mu(\rho)-1)d\big]\big\}\big\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}&.\end{aligned}\tag{7.10}$$

Applying product laws in Besov spaces ([6]) yields

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+c\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\leq\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+\sum_{j\in\mathbb{Z}}2^{j(-1+\frac{2}{p})}\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1_t(L^p)}&\\+C\|\rho-1\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})}\Big\{\|\partial_t u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|(u\cdot\nabla)u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\Big\}&.\end{aligned}\tag{7.11}$$

However, it following Lemma 7.1 and (7.8) that for $1<p<4$,

$$\|[\dot\Delta_j\mathbb{P};u\cdot\nabla]u\|_{L^1_t(L^p)}\lesssim d_j2^{j(1-\frac{2}{p})}\|\nabla u\|_{L^2_t(L^2)}^2\lesssim d_j2^{j(1-\frac{2}{p})}\|u_0\|_{L^2}^2,\tag{7.12}$$

and Lemma 7.2 together with (7.8) ensures that

$$\begin{aligned}\|(u\cdot\nabla)u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}&\lesssim\|u\|_{L^\infty_t(L^2)}\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}+\|\nabla u\|_{L^2_t(L^2)}\|u\|_{L^2_t(\dot B^{\frac{2}{p}}_{p,1})}\\&\lesssim\|u_0\|_{L^2}\big(\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\big).\end{aligned}\tag{7.13}$$

Substituting (7.12) and (7.13) into (7.11) results in

$$\begin{aligned}&\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+c\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\leq\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+C\|u_0\|_{L^2}^2\\&\quad+C(1+\|u_0\|_{L^2})\|\rho-1\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})}\Big\{\frac{\|\partial_t u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}}{1+\|u_0\|_{L^2}}+\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\Big\}.\end{aligned}\tag{7.14}$$

Whereas we infer from (7.9) and (7.13) that

$$\begin{aligned}&\|\partial_t u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|\nabla\Pi\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}\leq\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+C\|u_0\|_{L^2}\big(\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\big)\\&\quad+C(1+\|u_0\|_{L^2})\|\rho-1\|_{\widetilde L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})}\Big\{\frac{\|\partial_t u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}}{1+\|u_0\|_{L^2}}+\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\Big\}.\end{aligned}\tag{7.15}$$

For ε sufficiently small, we denote

$$\mathfrak{A}(t)\overset{\text{def}}{=}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}+\frac{\varepsilon}{1+\|u_0\|_{L^2}}\big(\|\partial_t u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}+\|\nabla\Pi\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}\big).\tag{7.16}$$

Then by summing (7.14) with $\frac{\varepsilon}{1+\|u_0\|_{L^2}}\times$(7.15), and using the following standard estimate on transport equation [6] that

$$\|\rho-1\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}\leq \|\rho_0-1\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\exp\bigl(C\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}}_{p,1})}\bigr),$$

we obtain

$$\mathfrak{A}(t)\leq C\bigl\{\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}+\|u_0\|_{L^2}^2+\mathfrak{A}(t)(1+\|u_0\|_{L^2})\|\rho_0-1\|_{\dot{B}^{\frac{2}{p}}_{p,1}}e^{C\mathfrak{A}(t)}\bigr\}. \tag{7.17}$$

In particular if we take ε_0 to be sufficiently small and C_0 to be sufficiently large in (1.11), one has

$$C\|\rho_0-1\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\exp\{2C(\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}+\|u_0\|_{L^2}^2)\}\leq\frac{1}{4},$$

which together with (7.17) ensures that

$$\mathfrak{A}(t)\leq 2C\bigl(\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}+\|u_0\|_{L^2}^2\bigr)\quad\text{for all}\quad t\in(0,T^*).$$

This in turn proves that $T^*=\infty$ under the assumption of (1.11), which completes the proof to the existence part of Theorem 1.3. □

To prove the uniqueness part of Theorem 1.3, we first recall the following Lemma from [13] (see Proposition 2.1 of [13]):

Lemma 7.3. *Let $v_0\in\dot{B}^s_{p,1}(\mathbb{R}^2)$ and $f\in L^1((0,T);\dot{B}^s_{p,1}(\mathbb{R}^2))$ with $p\in[1,\infty]$ and $s\in\mathbb{R}$. Let g,R satisfy $\nabla g\in L^1((0,T);\dot{B}^s_{p,1}(\mathbb{R}^2))$, $\partial_tR\in L^1((0,T);\dot{B}^s_{p,1}(\mathbb{R}^2))$ and that the compatibility condition $g|_{t=0}=\mathrm{div}v_0$ on $\mathbb{R}^2$. Then the system*

$$\begin{cases}\partial_tv-\Delta v+\nabla Q=f,\\ \mathrm{div}\,v=g=\mathrm{div}R,\\ v|_{t=0}=v_0\end{cases}$$

has a unique solution $(v,\nabla Q)$ so that

$$\begin{aligned}\|v\|_{\widetilde{L}^\infty_t(\dot{B}^s_{p,1})}&+\|(\partial_tv,\nabla^2v,\nabla Q)\|_{L^1_t(\dot{B}^s_{p,1})}\\ &\lesssim\|v_0\|_{\dot{B}^s_{p,1}}+\|f\|_{L^1_t(\dot{B}^s_{p,1})}+\|\nabla g\|_{L^1_t(\dot{B}^s_{p,1})}+\|\partial_tR\|_{L^1_t(\dot{B}^s_{p,1})}.\end{aligned} \tag{7.18}$$

Proof to the uniqueness part of Theorem 1.3. This part will essentially follow the Lagrangian idea from [13]. Yet in [13], the initial density belongs to the multiplier space $\mathcal{M}(B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d))$ of $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ for $1<p<2d$, and the viscosity coefficient $\mu(\rho)$ equals some positive constant. Here the initial density ρ_0 belongs to $B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$, which is a subspace of $\mathcal{M}(B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))$, but the viscous coefficient $\mu(\rho)$ depends on ρ. We remark that our proof here works in general space dimensions, although we only present here the 2-D case.

Let $(\rho,u,\nabla\Pi)$ be a global solution of (1.1) obtained in Theorem 1.3. Then as $u\in L^\infty(\mathbb{R}^+;Lip(\mathbb{R}^2))$, we can define the trajectory $X(t,y)$ of $u(t,x)$ by

$$\partial_tX(t,y)=u(t,X(t,y)),\qquad X(0,y)=y,$$

which leads to the following relation between the Eulerian coordinates x and the Lagrangian coordinates y:

$$x=X(t,y)=y+\int_0^tu(\tau,X(\tau,y))d\tau. \tag{7.19}$$

Moreover, we can take T so small that

$$\int_0^T\|\nabla u(t,\cdot)\|_{L^\infty}\leq\frac{1}{2}. \tag{7.20}$$

Then for $t \leq T$, $X(t,y)$ is invertible with respect to y variables, and we denote $Y(t,\cdot)$ to be its inverse mapping. Let

$$\bar{u}(t,y) \stackrel{\text{def}}{=} u(t,x) = u(t,X(t,y)) \quad \text{and} \quad \bar{\Pi}(t,y) \stackrel{\text{def}}{=} \Pi(t,X(t,y)).$$

Then similar to [13], one has

$$\bar{u} \in \widetilde{L}^\infty(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)) \quad \text{and} \quad \partial^2\bar{u}, \partial_t\bar{u}, \nabla\bar{\Pi} \in L^1(\mathbb{R}^+;\dot{B}_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)), \tag{7.21}$$

and

$$\begin{aligned} &\partial_t\bar{u}(t,y) = \partial_t u(t,x) + u(t,x)\nabla u(t,x),\\ &\partial_{x_i} u^j(t,x) = \partial_{y_k}\bar{u}^j(t,y)\partial_{x_i}y^k \quad \text{for} \quad x = X(t,y),\ y = Y(t,x) \end{aligned} \tag{7.22}$$

so that let $A(t,y) \stackrel{\text{def}}{=} (\nabla X(t,y))^{-1} = \nabla_x Y(t,x)$, we have

$$\nabla_x u(t,x) = A(t,y)^T\nabla_y\bar{u}(t,y) \quad \text{and} \quad \operatorname{div} u(t,x) = \operatorname{div}(A(t,y)\bar{u}(t,y)), \tag{7.23}$$

and $(\bar{u}, \nabla_y\bar{\Pi})$ solves

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}_y(\mu(\rho_0)d(\bar{u})) + \nabla_y\bar{\Pi} = \operatorname{div}\big(\mu(\rho_0)(AA^T - Id)d(\bar{u})\big) + (Id - A)^T\nabla_y\bar{\Pi},\\ \operatorname{div}\bar{u} = \operatorname{div}\big((Id - A)\bar{u}\big). \end{cases} \tag{7.24}$$

Now let $(\rho_i, u_i, \nabla\Pi_i)$, $i = 1,2$, be two solutions of (1.1) which satisfy the regularity properties listed in Theorem 1.3. Let $(\bar{u}_i, A_i, \bar{\Pi}_i)$, $i = 1,2$, be defined from (7.19) to (7.22), we denote

$$(\delta A, \delta\bar{u}, \nabla\delta\bar{\Pi}) \stackrel{\text{def}}{=} (A_2 - A_1, \bar{u}_2 - \bar{u}_1, \nabla\bar{\Pi}_2 - \nabla\bar{\Pi}_1).$$

Then it follows from (7.24) that the system for $(\delta\bar{u}, \nabla\delta\bar{\Pi})$ reads

$$\begin{cases} \partial_t\delta\bar{u} - \Delta_y\delta\bar{u} + \nabla_y\delta\bar{\Pi} = \delta\bar{F},\\ \operatorname{div}_y\delta\bar{u} = \nabla\delta\bar{u} : (Id - A_2) - \nabla u_1 : \delta A = \operatorname{div}_y\big((Id - A_2)\delta\bar{u} - \delta A\bar{u}_1\big),\\ \delta\bar{u}|_{t=0} = 0, \end{cases} \tag{7.25}$$

where

$$\begin{aligned} \delta\bar{F} =& (1-\rho_0)\partial_t\delta\bar{u} + \operatorname{div}_y[(\mu(\rho_0)-1)\nabla_y\delta\bar{u}] - \delta A^T\nabla_y\bar{\Pi}_1 + (Id - A_2)^T\nabla_y\delta\bar{\Pi}\\ &+ \operatorname{div}_y\big\{\mu(\rho_0)[(A_2A_2^T - Id)d(\delta\bar{u}) + (A_2A_2^T - A_1A_1^T)d(\bar{u}_1)]\big\}. \end{aligned}$$

We first deduce from product laws in Besov spaces ([6]) that

$$\begin{aligned} &\big\|(1-\rho_0)\partial_t\delta\bar{u} + \operatorname{div}_y[(\mu(\rho_0)-1)\nabla_y\delta\bar{u}]\big\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})}\\ &\qquad\lesssim \|(1-\rho_0)\|_{\dot{B}_{p,1}^{\frac{2}{p}}}\|\partial_t\delta\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|(\mu(\rho_0)-1)\|_{\dot{B}_{p,1}^{\frac{2}{p}}}\|\nabla_y\delta\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{\frac{2}{p}})}\\ &\qquad\lesssim \|(1-\rho_0)\|_{\dot{B}_{p,1}^{\frac{2}{p}}}\big(\|\partial_t\delta\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\delta\bar{u}\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{2}{p}})}\big). \end{aligned}$$

Before going further, we recall from [13, 14] that under the assumption of (7.20), one has

$$\begin{aligned} &\delta A(t) = \Big(\int_0^t \nabla\delta\bar{u}(\tau)\,d\tau\Big)\cdot\Big(\sum_{k\geq 1}\sum_{0\leq j\leq k} C_1^j(t)C_2^{k-1-j}(t)\Big),\\ &A_i(t) - Id = \sum_{k\geq 1}(-1)^k(C_i(t))^k \qquad \text{with} \qquad C_i(t) \stackrel{\text{def}}{=} \int_0^t \nabla\bar{u}_i(\tau)\,d\tau. \end{aligned} \tag{7.26}$$

Thanks to (7.26), for $p\in(1,4)$, we get, by applying product laws in Besov spaces ([6]), that

$$\begin{aligned}\big\|-\delta A^T\nabla\bar\Pi_1+(Id-A_2)^T\nabla\delta\bar\Pi\big\|_{L^1_t((\dot B^{-1+\frac2p}_{p,1})}&\lesssim\|\delta A\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\|\nabla\bar\Pi_1\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}+\|Id-A_2\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\|\nabla\delta\bar\Pi\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\lesssim\|\nabla\bar\Pi_1\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\|\nabla\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}+\|\nabla\bar u_2\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\nabla\delta\bar\Pi\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}.\end{aligned}$$

To deal with $\operatorname{div}\big\{\mu(\rho_0)(A_2A_2^T-Id)d(\delta\bar u)\big\}$, we write

$$\operatorname{div}\big\{\mu(\rho_0)(A_2A_2^T-Id)d(\delta\bar u)\big\}=\operatorname{div}\big\{\mu(\rho_0)[(A_2-Id)(A_2-Id)^T+A_2-Id+(A_2-Id)^T]d(\delta\bar u)\big\},$$

from which, we infer

$$\begin{aligned}&\big\|\operatorname{div}\big\{\mu(\rho_0)(A_2A_2^T-Id)d(\delta\bar u)\big\}\big\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\lesssim\big(1+\|\mu(\rho_0)-1\|_{\dot B^{\frac2p}_{p,1}}\big)\big(1+\|A_2-Id\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\big)\|A_2-Id\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\|\nabla\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\\&\lesssim\big(1+\|\nabla\bar u_2\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\big)\|\nabla\bar u_2\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\delta\bar u\|_{L^1_t(\dot B^{1+\frac2p}_{p,1})}.\end{aligned}$$

Similar estimate holds for $\operatorname{div}\big\{\mu(\rho_0)(A_2A_2^T-A_1A_1^T)d(\bar u_1)\big\}$. As a consequence, we obtain

$$\begin{aligned}\|\delta\bar F\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\lesssim&\big(\eta_1(t)+\|(1-\rho_0)\|_{\dot B^{\frac2p}_{p,1}}\big)\\&\times\Big(\|\partial_t\delta\bar u\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}+\|\delta\bar u\|_{L^1_t(\dot B^{1+\frac2p}_{p,1})}+\|\nabla\delta\bar\Pi\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\Big),\end{aligned}\tag{7.27}$$

with $\lim_{t\longrightarrow0}\eta_1(t)=0$.

On the other hand, we deduce from (7.25) and (7.26) that

$$\begin{aligned}\|\operatorname{div}\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\lesssim&\|\nabla\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|A_2-Id\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}+\|\nabla\bar u_1\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\delta A\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\\\lesssim&\big(\|\nabla\bar u_1\|_{L^1_t(\dot B^{\frac2p}_{p,1})}+\|\nabla\bar u_2\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\big)\|\delta\bar u\|_{L^1_t(\dot B^{1+\frac2p}_{p,1})}.\end{aligned}\tag{7.28}$$

Along the same line, we get

$$\begin{aligned}&\big\|\partial_t\big((Id-A_2)\delta\bar u-\delta A\bar u^1\big)\big\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\lesssim\|\nabla\bar u_2\delta\bar u\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}+\|(Id-A_2)\partial_t\delta\bar u\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}+\|\nabla\delta\bar u\bar u_1\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\quad+\Big\|\int_0^t|\nabla\delta\bar u\,dt'||\nabla\bar U_{1,2}||\bar u_1|\Big\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}+\Big\|\int_0^t|\nabla\delta u|\,dt'|\partial_t\bar u_1|\Big\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\lesssim\|\nabla\bar u_2\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\delta\bar u\|_{L^\infty_t(\dot B^{-1+\frac2p}_{p,1})}+\|Id-A_2\|_{L^\infty_t(\dot B^{\frac2p}_{p,1})}\|\partial_t\delta\bar u\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\quad+\|\nabla\delta\bar u\|_{L^2_t(\dot B^{-1+\frac2p}_{p,1})}\|\bar u_1\|_{L^2_t(\dot B^{\frac2p}_{p,1})}+\|\nabla\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\nabla\bar U_{1,2}\|_{L^2_t(\dot B^{-1+\frac2p}_{p,1})}\|\bar u_1\|_{L^2_t(\dot B^{\frac2p}_{p,1})}\\&\quad+\|\nabla\delta\bar u\|_{L^1_t(\dot B^{\frac2p}_{p,1})}\|\partial_t\bar u_1\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})},\end{aligned}$$

which implies

$$\begin{aligned}&\big\|\partial_t\big((Id-A_2)\delta\bar u-\delta A\bar u^1\big)\big\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\qquad\lesssim\eta_2(t)\Big(\|\delta\bar u\|_{L^\infty_t(\dot B^{-1+\frac2p}_{p,1})}+\|\delta\bar u\|_{L^1_t(\dot B^{1+\frac2p}_{p,1})}+\|\partial_t\delta\bar u\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\Big),\end{aligned}\tag{7.29}$$

where $\bar U_{1,2}$ denotes component of either $\bar u_1$ or $\bar u_2$, and $\lim_{t\longrightarrow0}\eta_2(t)=0$.

Thanks to Lemma 7.3, we get, by summing up (7.27) to (7.29), that

$$\begin{aligned}&\|\delta\bar u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac2p}_{p,1})}+\|(\partial_t\delta\bar u,\nabla^2\delta\bar u,\nabla\delta\bar\Pi)\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\\&\leq C(\eta(t)+\|\rho_0-1\|_{\dot B^{\frac2p}_{p,1}})\Big(\|\delta\bar u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac2p}_{p,1})}+\|(\partial_t\delta\bar u,\nabla^2\delta\bar u,\nabla\delta\bar\Pi)\|_{L^1_t(\dot B^{-1+\frac2p}_{p,1})}\Big),\end{aligned}\tag{7.30}$$

for some positive function $\eta(t)$ satisfying $\lim_{t\to0}\eta(t)=0$. Uniqueness part of Theorem 1.3 on a sufficiently small time interval $[0,t_1]$ follows (7.30). The whole time uniqueness then can be obtained by a bootstrap method. This completes the proof of Theorem 1.3. □

8. Proof of Theorem 1.4

In this section, we shall present the proof of Theorem 1.4 by following the same line to that of Theorem 1.3. For this, we first recall the following lemma from [1]:

Lemma 8.1. *[Lemma 2.1 of [1]] Let $s<\frac2p,(p,r)\in[1,\infty]^2$ be such that $s+2\inf\big(\frac1p,\frac1{p'}\big)>0$. Let $a\in\dot B^{\frac2p}_{p,\infty}\cap L^\infty(\mathbb R^2)$ and $b\in\dot B^s_{p,r}(\mathbb R^2)$. We denote $\lambda(s)\overset{def}{=}\frac{s+2\inf\big(\frac1p,\frac1{p'}\big)}{|s|+s+2\inf\big(\frac1p,\frac1{p'}\big)}$, then*

$$\|ab\|_{\dot B^s_{p,r}}\lesssim\|b\|_{\dot B^s_{p,r}}\begin{cases}\|a\|_{L^\infty}+\|a\|_{L^\infty}^{1-\max(0,s)\frac p2}\|a\|^{\max(0,s)\frac p2}_{\dot B^{\frac2p}_{p,\infty}}+\|a\|^{\lambda(s)}_{L^\infty}\|a\|^{1-\lambda(s)}_{\dot B^{\frac2p}_{p,\infty}} & \text{if } s\neq0\\ \|a\|_{L^\infty}\ln\big(e+\|a\|_{\dot B^{\frac2p}_{p,\infty}}\|a\|^{-1}_{L^\infty}\big) & \text{if } s=0.\end{cases}$$

Lemma 8.2. *Let $p\in(1,\infty),0<\varepsilon<\frac2p$. Then for $a\in\dot B^{\frac2p}_{p,\infty}\cap\dot B^{\frac2p+\varepsilon}_{p,\infty}\cap L^\infty(\mathbb R^2)$ and $b\in\dot B^{\frac2p-\varepsilon}_{p,\infty}\cap\dot B^{\frac2p}_{p,\infty}(\mathbb R^2)$, one has*

$$\begin{aligned}&\|ab\|_{\dot B^{\frac2p}_{p,1}}\lesssim\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}+\|a\|^{\frac{p\varepsilon}{2+p\varepsilon}}_{L^\infty}\|a\|^{\frac{2}{2+p\varepsilon}}_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|^{\frac{2}{2+p\varepsilon}}_{\dot B^{\frac2p-\varepsilon}_{p,1}}\|b\|^{\frac{p\varepsilon}{2+p\varepsilon}}_{\dot B^{\frac2p}_{p,1}},\\&\|ab\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\lesssim\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}+\|a\|^{\frac{2\varepsilon}{p}}_{L^\infty}\|a\|^{1-\frac{2\varepsilon}{p}}_{\dot B^{\frac2p}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}.\end{aligned}\tag{8.1}$$

Proof. Bony's decomposition (A.5) for ab reads

$$ab=T_ab+T_ba+\mathcal R(a,b).$$

It follows from para-product estimate that

$$\|T_ab+\mathcal R(a,b)\|_{\dot B^{\frac2p}_{p,1}}\lesssim\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}\quad\text{and}\quad\|T_ab+\mathcal R(a,b)\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\lesssim\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}.\tag{8.2}$$

To deal with T_ba, for any integer $M>0$, we write

$$\|\dot\Delta_j(T_ba)\|_{L^p}\lesssim\sum_{|\ell-j|\leq4}\Big\{\|\dot\Delta_\ell a\|_{L^p}\sum_{k\leq\ell-M}\|\dot\Delta_kb\|_{L^\infty}+\|\dot\Delta_\ell a\|_{L^\infty}\sum_{\ell-M<k\leq\ell}\|\dot\Delta_kb\|_{L^p}\Big\},$$

so that

$$\begin{aligned}\|T_ba\|_{\dot B^{\frac2p}_{p,1}}\lesssim&\sum_{\substack{\ell\in\mathbb Z\\k\leq\ell-M}}2^{(k-\ell)\varepsilon}d_k\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}+\sum_{\substack{\ell\in\mathbb Z\\\ell-M<k\leq\ell}}2^{(\ell-k)\frac2p}d_k\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}\\\lesssim&2^{-M\varepsilon}\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}+2^{\frac{2M}{p}}\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}.\end{aligned}$$

Thus choosing $M=\Big[\frac{p}{2+p\varepsilon}\log_2\frac{\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}}{\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}}\Big]$ in the above inequality gives rise to

$$\|T_ba\|_{\dot B^{\frac2p}_{p,1}}\lesssim\big(\|a\|_{L^\infty}\|b\|_{\dot B^{\frac2p}_{p,1}}\big)^{\frac{p\varepsilon}{2+p\varepsilon}}\big(\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\big)^{\frac{2}{2+p\varepsilon}}. \tag{8.3}$$

This together with (8.2) proves the first inequality of (8.1).

Along the same line to proof of (8.3), for any positive integer M, one has

$$\|T_ba\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\lesssim\big(2^{-M\varepsilon}\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}+2^{(\frac2p-\varepsilon)M}\|a\|_{L^\infty}\big)\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}.$$

Choosing $M=\Big[\frac2p\log_2\frac{\|a\|_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}}{\|a\|_{L^\infty}}\Big]$ in the above inequality leads to

$$\|T_ba\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\lesssim\|a\|_{L^\infty}^{\frac{2\varepsilon}{p}}\|a\|^{1-\frac{2\varepsilon}{p}}_{\dot B^{\frac2p+\varepsilon}_{p,\infty}}\|b\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}.$$

This together with (8.2) proves the second inequality of (8.1). $\square$

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. The proof of Theorem 1.4 essentially follows from that of Theorem 1.3. For simplicity, we just present the *a priori* estimates for smooth enough solution $(\rho,u,\nabla\Pi)$ of (1.1). We first get, by a similar derivation (7.10), that

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac2p-\varepsilon}_{p,1})}&+c\|u\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}\\&\lesssim\|u_0\|_{\dot B^{-1+\frac2p-\varepsilon}_{p,1}}+\sum_{j\in\mathbb Z}2^{j(-1+\frac2p-\varepsilon)}\|[\dot\Delta_j\mathbb P;u\cdot\nabla]u\|_{L^1_t(L^p)}\\&\quad+\|(1-\rho)\big(\partial_tu+u\cdot\nabla u\big)\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}+\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot B^{\frac2p-\varepsilon}_{p,1})}.\end{aligned}$$

However, it follows from Lemma 7.1 that

$$\begin{aligned}\|[\dot\Delta_j\mathbb P;u\cdot\nabla]u\|_{L^1_t(L^p)}\lesssim&d_j2^{j(1-\frac2p+\varepsilon)}\int_0^t\|\nabla u(t')\|_{L^2}\|u(t')\|_{\dot B^{\frac2p-\varepsilon}_{p,1}}\,dt'\\\lesssim&d_j2^{j(1-\frac2p+\varepsilon)}\int_0^t\|\nabla u(t')\|_{L^2}\|u(t')\|^{\frac12}_{\dot B^{-1+\frac2p-\varepsilon}_{p,1}}\|u(t')\|^{\frac12}_{\dot B^{1+\frac2p-\varepsilon}_{p,1}}\,dt',\end{aligned}$$

so that one has

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac2p-\varepsilon}_{p,1})}&+c\|u\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}\\&\lesssim C\Big\{\|u_0\|_{\dot B^{-1+\frac2p-\varepsilon}_{p,1}}+\|(1-\rho)\big(\partial_tu+u\cdot\nabla u\big)\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}+\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot B^{\frac2p-\varepsilon}_{p,1})}\\&\quad+\int_0^t\|\nabla u(t')\|_{L^2}^2\|u(t')\|_{\dot B^{-1+\frac2p-\varepsilon}_{p,1}}\,dt'\Big\}+\frac c2\|u\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}.\end{aligned}$$

Applying Gronwall's inequality gives rise to

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^{-1+\frac2p-\varepsilon}_{p,1})}+c\|u\|_{L^1_t(\dot B^{1+\frac2p-\varepsilon}_{p,1})}\leq C\exp\big(C\|u_0\|_{L^2}^2\big)\Big\{\|u_0\|_{\dot B^{-1+\frac2p-\varepsilon}_{p,1}}&\\+\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot B^{\frac2p-\varepsilon}_{p,1})}+\|(1-\rho)\big(\partial_tu+u\cdot\nabla u\big)\|_{L^1_t(\dot B^{-1+\frac2p-\varepsilon}_{p,1})}\Big\}.&\end{aligned} \tag{8.4}$$

Whereas we deduce, by a similar derivation of (7.15), that

$$
\begin{aligned}
&\|\partial_t u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|\nabla\Pi\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\\
&\leq\|u_0\|_{\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+C\Big\{\|u_0\|_{L^2}\big(\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1})}\big)\\
&\qquad+\|(1-\rho)\big(\partial_t u+u\cdot\nabla u\big)\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot{B}^{\frac{2}{p}-\varepsilon}_{p,1})}\Big\}.
\end{aligned}
\tag{8.5}
$$

For ε sufficiently small, we denote

$$
\mathfrak{A}_\varepsilon(t)\stackrel{\text{def}}{=}\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1})}+\frac{\varepsilon}{1+\|u_0\|_{L^2}}\Big(\|\partial_t u\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|\nabla\Pi\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\Big).
$$

Then summing (8.4) with $\frac{\varepsilon}{1+\|u_0\|_{L^2}}\times$(8.5), we obtain

$$
\begin{aligned}
\mathfrak{A}_\varepsilon(t)\leq &C\exp\big\{C\|u_0\|_{L^2}^2\big\}\Big\{\|u_0\|_{\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot{B}^{\frac{2}{p}-\varepsilon}_{p,1})}\\
&\qquad+\|(1-\rho)\big(\partial_t u+u\cdot\nabla u\big)\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\Big\}.
\end{aligned}
\tag{8.6}
$$

Similar estimate holds for $\mathfrak{A}(t)$ defined by (7.16).

On the other hand, without loss of generality, we may assume that $-1+\frac{2}{p}-\varepsilon\neq 0$, applying (7.13) and Lemma 8.1 that

$$
\begin{aligned}
&\|(1-\rho)\big(\partial_t u+u\cdot\nabla u\big)\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\\
&\lesssim\Big(\|\rho-1\|_{L^\infty_t(L^\infty)}+\|\rho-1\|_{L^\infty_t(L^\infty)}^{1-\max(0,-1+\frac{2}{p}-\varepsilon)\frac{p}{2}}\|\rho-1\|_{L^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}^{\max(0,-1+\frac{2}{p}-\varepsilon)\frac{p}{2}}\\
&\qquad+\|\rho-1\|_{L^\infty_t(L^\infty)}^{\lambda(-1+\frac{2}{p}-\varepsilon)}\|\rho-1\|_{L^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}^{1-\lambda(-1+\frac{2}{p}-\varepsilon)}\Big)\|(\partial_t u+u\cdot\nabla u)\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\\
&\lesssim\|\rho_0-1\|_{L^\infty}^{\theta(\varepsilon)}\|\rho-1\|_{L^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}^{1-\theta(\varepsilon)}\Big(\|\partial_t u\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}\\
&\qquad+\|u_0\|_{L^2}(\|u\|_{\widetilde{L}^\infty_t(\dot{B}^{-1+\frac{2}{p}-\varepsilon}_{p,1})}+\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1})})\Big),
\end{aligned}
$$

where $\lambda(-1+\frac{2}{p}-\varepsilon)$ is given by Lemma 8.1 and $\theta(\varepsilon)\stackrel{\text{def}}{=}\min\big\{1-\max(0,-1+\frac{2}{p}-\varepsilon)\frac{p}{2},\lambda(-1+\frac{2}{p}-\varepsilon)\big\}$. Similar estimate holds for $\|(1-\rho)\big(\partial_t u+u\cdot\nabla u\big)\|_{L^1_t(\dot{B}^{-1+\frac{2}{p}}_{p,1})}$.

While we get, by applying Lemma 8.2, that

$$
\begin{aligned}
\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot{B}^{\frac{2}{p}-\varepsilon}_{p,1})}\lesssim&\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1})}\Big(\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}\\
&\qquad+\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}^{\frac{2\varepsilon}{p}}\|\mu(\rho)-1\|_{L^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}^{1-\frac{2\varepsilon}{p}}\Big)\\
\lesssim&\|\rho_0-1\|_{L^\infty}^{\frac{2\varepsilon}{p}}\|\rho-1\|_{L^\infty_t(\dot{B}^{\frac{2}{p}}_{p,1})}^{1-\frac{2\varepsilon}{p}}\|u\|_{L^1_t(\dot{B}^{1+\frac{2}{p}-\varepsilon}_{p,1})},
\end{aligned}
$$

and

$$\begin{aligned}\|(\mu(\rho)-1)d(u)\|_{L^1_t(\dot B^{\frac{2}{p}}_{p,1})} \lesssim\; &\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\\
&+\big(\|\mu(\rho)-1\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\big)^{\frac{p\varepsilon}{2+p\varepsilon}}\big(\|\mu(\rho)-1\|_{L^\infty_t(\dot B^{\frac{2}{p}+\varepsilon}_{p,1})}\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}-\varepsilon}_{p,1})}\big)^{\frac{2}{2+p\varepsilon}}\\
\lesssim&\|\rho_0-1\|^{\frac{p\varepsilon}{2+p\varepsilon}}_{L^\infty}\big(\|\rho_0-1\|^{\frac{2}{2+p\varepsilon}}_{L^\infty}+\|\rho-1\|^{\frac{2}{2+p\varepsilon}}_{L^\infty_t(\dot B^{\frac{2}{p}+\varepsilon}_{p,1})}\big)\Big(\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}-\varepsilon}_{p,1})}+\|u\|_{L^1_t(\dot B^{1+\frac{2}{p}}_{p,1})}\Big).\end{aligned}$$

Therefore, we deduce from (8.6) that

$$\begin{aligned}\mathfrak{A}(t)+\mathfrak{A}_\varepsilon(t)\le& C\exp\big(C\|u_0\|^2_{L^2}\big)\Big\{\|u_0\|_{\dot B^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+\|\rho_0-1\|^{\delta(\varepsilon)}_{L^\infty}\big[1\\
&+\|\rho-1\|_{L^\infty_t(\dot B^{\frac{2}{p}}_{p,1})}+\|\rho-1\|_{L^\infty_t(\dot B^{\frac{2}{p}+\varepsilon}_{p,1})}\big]\big(\mathfrak{A}(t)+\mathfrak{A}_\varepsilon(t)\big)\Big\}\\
\le& C\exp\big(C\|u_0\|^2_{L^2}\big)\Big\{\|u_0\|_{\dot B^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}+\|\rho_0-1\|^{\delta(\varepsilon)}_{L^\infty}\big[1\\
&+\|\rho_0-1\|_{\dot B^{\frac{2}{p}}_{p,1}}+\|\rho_0-1\|_{\dot B^{\frac{2}{p}+\varepsilon}_{p,1}}\big]\big(\mathfrak{A}(t)+\mathfrak{A}_\varepsilon(t)\big)\exp\big(C\mathfrak{A}(t)\big)\Big\}.\end{aligned}\tag{8.7}$$

In particular, if $\|\rho_0-1\|_{L^\infty}$ is so small that

$$\begin{aligned}&C\|\rho_0-1\|^{\delta(\varepsilon)}_{L^\infty}\big(1+\|\rho_0-1\|_{\dot B^{\frac{2}{p}}_{p,1}}+\|\rho_0-1\|_{\dot B^{\frac{2}{p}+\varepsilon}_{p,1}}\big)\\
&\quad\times\exp\big(C\|u_0\|^2_{L^2}\big)\exp\Big\{2C\exp\big(C\|u_0\|^2_{L^2}\big)\big(\|u_0\|_{\dot B^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}\big)\Big\}\le\frac{1}{2},\end{aligned}\tag{8.8}$$

for $\delta(\varepsilon)\stackrel{\text{def}}{=}\min\big(\theta(0),\theta(\varepsilon),\frac{2\varepsilon}{p},\frac{p\varepsilon}{2+p\varepsilon}\big)$, we infer from (8.7) that

$$\mathfrak{A}(t)+\mathfrak{A}_\varepsilon(t)\le 2C\exp\big(C\|u_0\|^2_{L^2}\big)\Big(\|u_0\|_{\dot B^{-1+\frac{2}{p}-\varepsilon}_{p,1}}+\|u_0\|_{\dot B^{-1+\frac{2}{p}}_{p,1}}\Big).$$

With this *a priori* estimate, we complete the proof to the existence part of Theorem 1.4. The proof to the uniqueness part is identical to that of Theorem 1.3. One only needs to use Lemma 8.1 and Lemma 8.2 rather than the standard product laws to estimate $\|(1-\rho_0)\partial_t\delta\bar u\|_{L^1_t(\dot B^{-1+\frac{2}{p}}_{p,1})}$ and $\|(\mu(\rho_0)-1)\nabla_y\bar u\|_{L^1_t(\dot B^{\frac{2}{p}}_{p,1})}$. We skip the details here. □

Appendix A. Littlewood-Paley analysis

The proofs of Theorem 1.2 to Theorem 1.4 requires Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^d$ (see e.g. [6]). Let φ be a smooth function supported in the ring $\mathcal{C}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^d,\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and $\chi(\xi)$ be a smooth function supported in the ball $\mathcal{B}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^d,\ |\xi|\le\frac{4}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for}\quad\xi\neq0\quad\text{and}\quad\chi(\xi)+\sum_{q\ge0}\varphi(2^{-q}\xi)=1\quad\text{for all}\quad\xi\in\mathbb{R}^d.$$

Then for $u\in\mathcal{S}'_h(\mathbb{R}^d)$ (see Definition 1.26 of [6]), which means $u\in\mathcal{S}'(\mathbb{R}^d)$ and $\lim_{j\to-\infty}\|\chi(2^{-j}D)u\|_{L^\infty}=0$, we set

$$\begin{aligned}&\forall\ j\in\mathbb{Z},\quad\dot\Delta_ju\stackrel{\text{def}}{=}\varphi(2^{-j}\mathrm{D})u\quad\text{and}\quad\dot S_ju\stackrel{\text{def}}{=}\chi(2^{-j}D)u,\\
&\forall\ q\ge0,\quad\Delta_qu\stackrel{\text{def}}{=}\varphi(2^{-q}\mathrm{D})u,\quad\Delta_{-1}u\stackrel{\text{def}}{=}\chi(D)u\quad\text{and}\quad S_qu\stackrel{\text{def}}{=}\sum_{-1\le q'\le q-1}\Delta_{q'}u,\end{aligned}\tag{A.1}$$

we have the formal decomposition

$$u=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j\,u,\quad \forall\,u\in\mathcal{S}'_h(\mathbb{R}^d)\quad\text{and}\quad u=\sum_{q\geq-1}\Delta_q\,u\quad\forall\,u\in\mathcal{S}(\mathbb{R}^d).\tag{A.2}$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j\dot{\Delta}_q u\equiv0\quad\text{if}\quad|j-q|\geq2\quad\text{and}\quad\dot{\Delta}_j(\dot{S}_{q-1}u\dot{\Delta}_q u)\equiv0\quad\text{if}\quad|j-q|\geq5.\tag{A.3}$$

We recall now the definition of homogeneous Besov spaces and Bernstein type inequalities from [6]. Similar definitions in the inhomogeneous context can be found in [6].

Definition A.1. *[Definition 2.15 of [6]] Let $(p,r)\in[1,+\infty]^2$, $s\in\mathbb{R}$ and $u\in\mathcal{S}'_h(\mathbb{R}^3)$, we set*

$$\|u\|_{\dot{B}^s_{p,r}}\stackrel{\text{def}}{=}\Big(2^{js}\|\dot{\Delta}_j u\|_{L^p}\Big)_{\ell^r}.$$

- *For $s<\frac{3}{p}$ (or $s=\frac{3}{p}$ if $r=1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^3)\stackrel{\text{def}}{=}\big\{u\in\mathcal{S}'_h(\mathbb{R}^3)\;\big|\;\|u\|_{\dot{B}^s_{p,r}}<\infty\big\}$.*
- *If $k\in\mathbb{N}$ and $\frac{3}{p}+k\leq s<\frac{3}{p}+k+1$ (or $s=\frac{3}{p}+k+1$ if $r=1$), then $\dot{B}^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u\in\mathcal{S}'_h(\mathbb{R}^3)$ such that $\partial^\beta u\in\dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta|=k$.*

Lemma A.1. *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^d$. A constant C exists so that for any positive real number δ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers $(a,\ b)$ with $b\geq a\geq1$, there hold*

$$\begin{aligned}&\operatorname{Supp}\ \hat{u}\subset\delta\mathcal{B}\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^{k+1}\delta^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\&\operatorname{Supp}\ \hat{u}\subset\delta\mathcal{C}\Rightarrow C^{-1-k}\delta^k\|u\|_{L^a}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^{1+k}\delta^k\|u\|_{L^a},\\&\operatorname{Supp}\ \hat{u}\subset\delta\mathcal{C}\Rightarrow\|\sigma(D)u\|_{L^b}\leq C_{\sigma,m}\delta^{m+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.\end{aligned}\tag{A.4}$$

We also recall Bony's decomposition from [7]:

$$uv=T_uv+R(u,v)=T_uv+T_vu+\mathcal{R}(u,v),\tag{A.5}$$

where

$$\begin{aligned}&T_uv\stackrel{\text{def}}{=}\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_jv,\qquad R(u,v)\stackrel{\text{def}}{=}\sum_{j\in\mathbb{Z}}\dot{\Delta}_ju\dot{S}_{j+2}v,\\&\mathcal{R}(u,v)\stackrel{\text{def}}{=}\sum_{j\in\mathbb{Z}}\dot{\Delta}_ju\dot{\tilde{\Delta}}_jv\quad\text{with}\quad\dot{\tilde{\Delta}}_jv\stackrel{\text{def}}{=}\sum_{|j'-j|\leq1}\dot{\Delta}_{j'}v.\end{aligned}$$

To prove Theorem 4.1, we need the following lemma concerning the commutator estimates, the proof of which is a standard application of Basic Littlewood-Paley theory.

Lemma A.2. *Let $s>0$, $a\in\dot{H}^{1+s}\cap Lip(\mathbb{R}^2)$ and $b\in L^\infty\cap Lip\cap H^s(\mathbb{R}^2)$. Then there holds*

$$\begin{aligned}&\|[\dot{\Delta}_j;a]\nabla b\|_{L^2}\lesssim c_j2^{-js}\big(\|\nabla a\|_{L^\infty}\|b\|_{\dot{H}^s}+\|\nabla b\|_{L^\infty}\|a\|_{\dot{H}^s}\big),\\&\|[\dot{\Delta}_j;a]\nabla b\|_{L^2}\lesssim c_j2^{-js}\big(\|\nabla a\|_{L^\infty}\|b\|_{\dot{H}^s}+\|b\|_{L^\infty}\|a\|_{\dot{H}^{1+s}}\big),\\&\|[\dot{\Delta}_j;a]\nabla b\|_{L^2}\lesssim c_j2^{-js}\big(\|\nabla a\|_{L^\infty}\|b\|_{\dot{H}^s}+\|a\|_{\dot{H}^{1+s}}\|\nabla b\|_{L^2}\big).\end{aligned}$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r}(\mathbb{R}^d))$ from [6].

Definition A.2. *Let $(r,\lambda,p)\in[1,+\infty]^3$ and $T\in]0,+\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p\,r}(\mathbb{R}^d))$ as the completion of $C([0,T];\,\mathcal{S}(\mathbb{R}^d))$ by the norm*

$$\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})}\stackrel{def}{=}\Big(\sum_{j\in\mathbb{Z}}2^{jrs}\Big(\int_0^T\|\dot{\Delta}_j\,f(t)\|^\lambda_{L^p}\,dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}}<\infty.$$

with the usual change if $r=\infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$.

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